Panel 1

Last Time: How to find Relative Extremas

① ∇f

② Solve $\nabla f = 0$

③ $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \Rightarrow D = f_{xx} f_{yy} - (f_{xy})^2$

④ $D > 0$, $f_{xx} > 0$, min

$D > 0$, $f_{xx} < 0$: max

$D < 0$, saddle

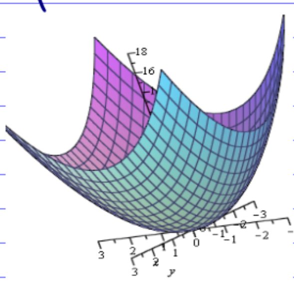
$D = 0$ no info

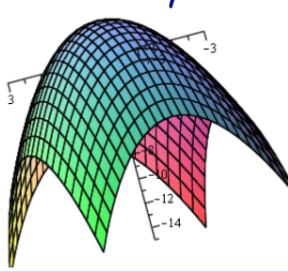
$f_{xx} = 0 \Rightarrow$ saddle point or no info

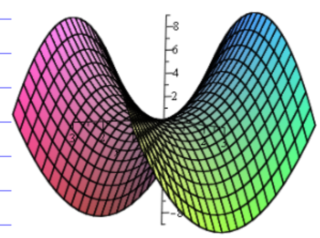
1

Panel 2

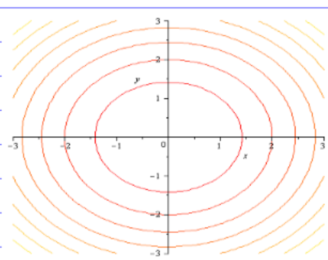
$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $H = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ $H = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

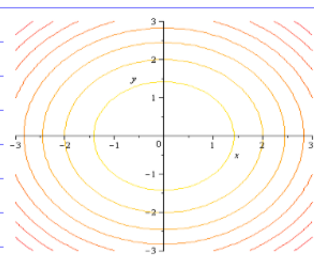


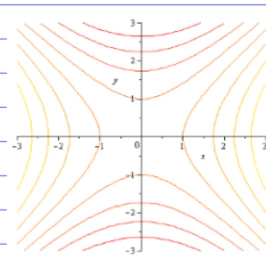




$f(x,y) = x^2 + y^2$ $f(x,y) = 1 - x^2 - y^2$ $f(x,y) = x^2 - y^2$







2

Panel 3

Ex: Find and classify the critical points for $f(x,y) = x^3y + 12x^2 - 8y$

$f_x = 3x^2y + 24x = 0 \qquad \underbrace{3x}\,\underbrace{(xy+8)} = 0 \qquad , \cancel{x=0}$ no good in 2^{nd} eqn.

$f_y = x^3 - 8 = 0 \qquad \Rightarrow \underline{x = 2}, \ \underline{y = -4}$ because $xy+8=0$

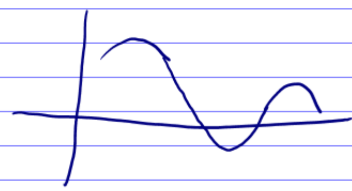
$$H = \begin{pmatrix} 6xy+24 & 3x^2 \\ 3x^2 & 0 \end{pmatrix}, \quad D = -9x^4 < 0 \text{ at } (2,-4).$$

$\Rightarrow (2,-4)$ is a saddle

3

Panel 4

$f: \mathbb{R} \to \mathbb{R}$ has 2 local max $\Rightarrow$ f has at least 1 min



$f(1,2)=0$

$f(x,y) = -(x^2-1)^2 - (x^2y-x-1)^2 \leq 0$ but $f(-1,0)=0$ $\Rightarrow$ $\underline{max}$

$f_x = \underline{-2(x^2-1)2x} - 2(x^2y-x-1)(2xy-1) = 0$

$f_y = -\underline{2(x^2y-x-1)}\underline{(x^2)} = 0$

if $x=0$ X if $x^2y-x-1=0 \Rightarrow \begin{pmatrix} x=+1 \\ y=2 \\ \underline{max} \end{pmatrix}, \begin{pmatrix} \text{or } x=-1 \\ y=0 \\ \underline{D=0} \end{pmatrix}$

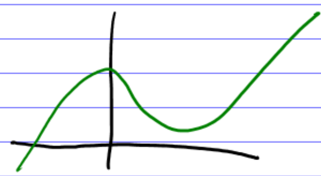
4

Panel 5

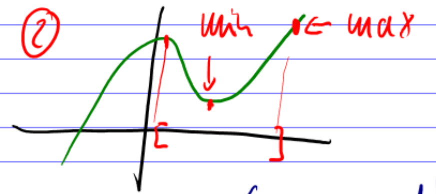
Absolute Max / Min:

Differences between absolute and relative extrema

(1)



(2)



(compact)

<u>Thm:</u> If f is continuous on a closed, bounded set, then f has abs. max and abs. min. Moreover the extrema can occur only at critical points or <u>on the boundary</u>

5

Panel 6

<u>To find abs. max/min</u>

If f is continuous on closed bounded set $D \subset \mathbb{R}^2$:

(1) $\nabla f = 0$

(2) Find all critical point on bdry of D

(3) Check endpoints if any

(4) Pick largest/smallest values

rel. max/min

(1) $\nabla f = 0$

(2) H, D, f_{xx}

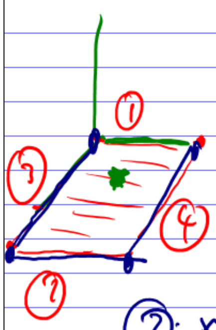
6

Panel 7

Ex: Find abs. extrema for $f(x,y) = x^2 - 2xy + 2y$ on $[0,3]\times[0,2]$, i.e. $0 \le x \le 3$ and $0 \le y \le 2$

$f_x = 2x - 2y = 0$ $\quad y = 1$

$f_y = -2x + 2 = 0$, $\quad x = 1$



(1): $x = 0$, $y \in [0,2]$: $f(0,y) = 2y$ no extra critical points

(2): $x = 3$, $y \in [0,2]$: $f(3,y) = 9 - 4y$ no extra criticals

(3) $y = 0$, $x \in [0,3]$: $f(x,0) = x^2$ ⇒ critical $x = 0$ $(y = 0)$

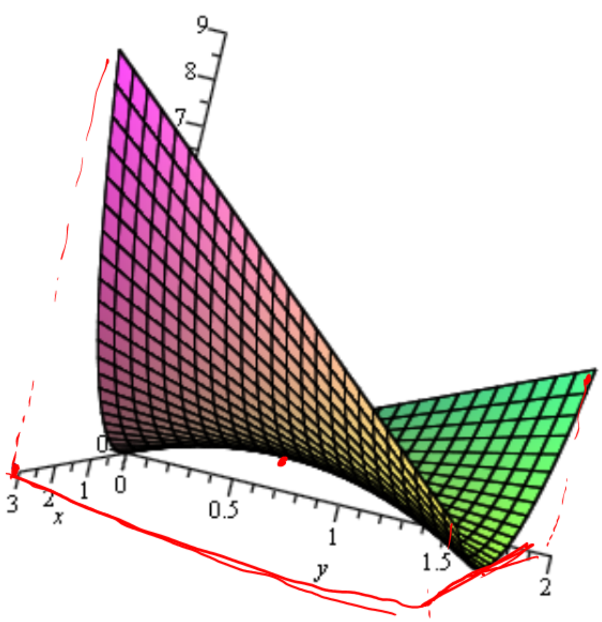
(4) $y = 2$, $x \in [0,3]$: $f(x,0) = x^2 - 4x + 4$ ⇒ critical $x = 2$ $(y = 2)$

Check: $f(1,1) =$, $f(2,2)$, $f(0,0)$

$f(0,0) =$ $\quad f(3,0) =$ $\quad f(3,2) =$ $\quad f(0,2) =$

7

Panel 8



(x,y)	$z = f(x,y)$
$(1,1)$	1
$(2,2)$	0
$(0,0)$	0
$(0,2)$	4
$(3,0)$	9
$(3,2)$	2

min

max

8

Panel 9

$f(x,y) = x^2 + 2y^2 + 4xy$ for $(x,y) \in [0,1]\times[0,1]$. → abs. extrema?

$f_x = 2x + 4y = 0$ ⇒ $(x,y) = (0,0)$

$f_y = 4y + 4x = 0$

Let $x=0,\ y\in[0,1]$: $f(0,y) = 2y^2$ ⇒ $y=0$

Let $x=1,\ y\in[0,1]$: $f(1,y) = 1+2y^2+4y$ ⇒ $\cancel{y=-1}$

Let $y=0,\ x\in[0,1]$: $f(x,0) = x^2$ ⇒ $x=0$

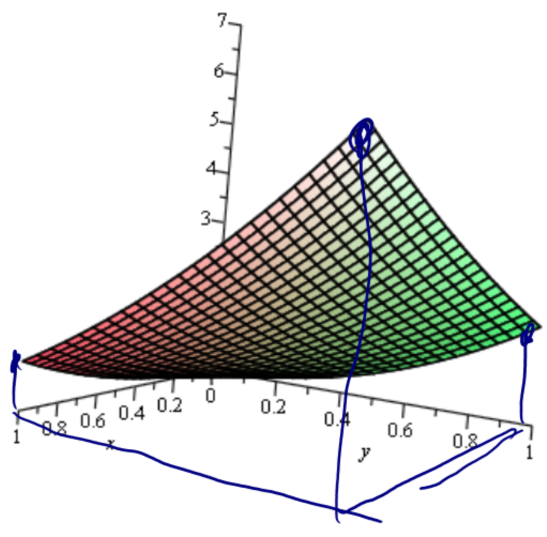
Let $y=1,\ x\in[0,1]$: $f(x,1) = x^2+2+4x$ ⇒ $\cancel{x=-2}$

	(x,y)	$f(x,y)$
min	$(0,0)$	0
	$(0,1)$	2
	$(1,0)$	1
max	$(1,1)$	7

9

Panel 10

$f(x,y) = x^2 + 2y^2 + 4xy$

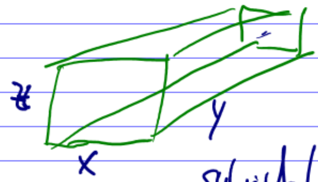


(x,y)	$f(x,y) = z$
$(0,0)$	
$(0,1)$	
$(1,0)$	
$(1,1)$	

10

Panel 11

Ex: Make a box w/o lid out of 12 cm² cardboard of max volume.



$V = xyz$ max.

subject to $2xz + 2yz + xy = 12$

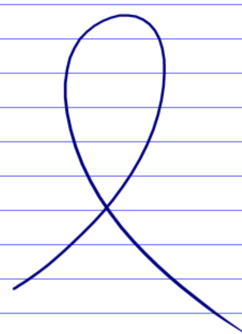
$z(2x+2y) = 12 - xy$

$z = \frac{12-xy}{2x+2y}$

$V = xy\,\frac{12-xy}{2x+2y}$

Panel 12

Problem: Max $P = xyz$ subject to $2xz + 2yz + xy = 12$



use Maple

Panel 13

Problem: Max $V = xyz$ subject to $2xz + 2yz + xy = 12$

$\Rightarrow V(x,y) = \dfrac{xy(12-xy)}{2x+2y}$, $(x,y) \in [0,\infty) \times [0,\infty)$

$V(x,y) := \dfrac{x\,y\,(12-x\,y)}{2\,x+2\,y}$

$$(x,y) \rightarrow \frac{x\,y\,(12-x\,y)}{2\,x+2\,y} \quad (1)$$

$Vx := diff(V(x,y), x)$

$$\frac{y\,(12-x\,y)}{2\,x+2\,y} - \frac{x\,y^2}{2\,x+2\,y} - \frac{2\,x\,y\,(12-x\,y)}{(2\,x+2\,y)^2} \quad (2)$$

$Vy := diff(V(x,y), y);$

$$\frac{x\,(12-x\,y)}{2\,x+2\,y} - \frac{x^2\,y}{2\,x+2\,y} - \frac{2\,x\,y\,(12-x\,y)}{(2\,x+2\,y)^2} \quad (3)$$

$solve(\{Vx = 0, Vy = 0\}, \{x, y\})$

$$\{x = 2, y = 2\}, \{x = -2, y = -2\} \quad (4)$$

$x=2, y=2$

got to be a max.

13

Panel 14

Other Method: Lagrange Multipliers

Maximize $f(x,y,z)$ subject to $g(x,y,z) = k$

next time

Quiz on Wed on abs max/min

14