Panel 1

Last Time $f(x,y)$ $x=f(t)$, $y=g(t)$

Chain Rule: $\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$

Directional Derivative: $D_{\vec{v}}(f) = \langle f_x, f_y\rangle \cdot \vec{v}$ $\|\vec{v}\|=1$

Gradient: $\nabla f = \langle f_x, f_y \rangle$

Properties of Gradient:

1

Panel 2

Properties of Gradient

- The gradient is a vector
- Gradient is perpendicular to level curves
- Gradient points in direction of steepest increase
- $\|\nabla f\|$ is the max. rate of change

Ex: Find ∇f if $f(x,y,z) = \ln(xy^2z^3)$

$\nabla f = \left\langle \frac{1 \cdot y^2z^3}{xy^2z^3}, \frac{1}{1xy^2z^3} 2yxz^3, \frac{1}{xy^2z^3} 3z^2xy^2 \right\rangle$

Find max. rate of change for $f(x,y,z) = \ln(xy^2z^3)$ at $P(1,1,1)$

$\nabla f = \langle \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \rangle$, $\|\nabla f\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

2

Panel 3

Ex: Suppose the level curves of an area are given by $f(x,y) = y\ln(x)$. You are standing at $P(1,-3)$ and you are heading in the direction $\langle -4, 3\rangle$.

a) Are you going up or down? How much? b) Which way should you go for max change in height?

$D_v(f)$ at $P(1,-3)$, where $v = \frac{1}{5}\langle -4, 3\rangle$

$\langle f_x, f_y\rangle = \langle \frac{y}{x}, \ln(x)\rangle$, at $P(1,-3) \Rightarrow \nabla f\big|_{(1,-3)} = \langle -3, 0\rangle$

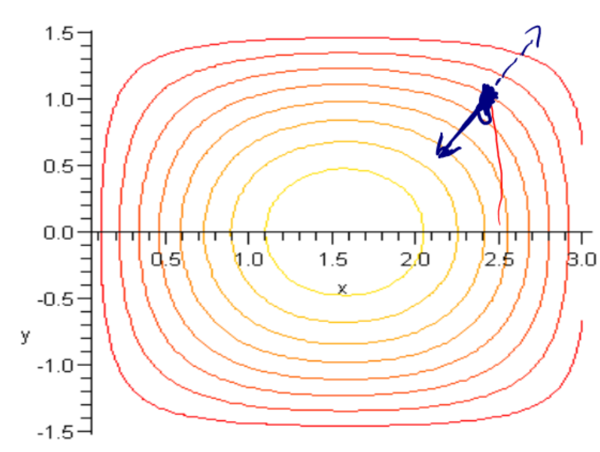
a) $D_v(f) = \langle -3, 0\rangle \cdot \frac{1}{5}\langle -4, 3\rangle = \frac{1}{5}\, 12 = \frac{12}{5} \rightarrow$ up

b) $\nabla f = \langle -3, 0\rangle$ for max rate of change of $3 = \|\nabla f\|$

3

Panel 4

Below is a contour plot for $f(x,y)$, showing several level curves. Sketch ∇f at $P(2.5, 1.0)$, approx.



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Panel 5

$f(x,y,z) = \sqrt{x+yz} = (x+yz)^{1/2}$, $P(1,3,1)$

$f_x = \frac{1}{2}(x+yz)^{-1/2} = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

$f_y = \frac{1}{2}(x+yz)^{-1/2}\cdot z = \frac{1}{2}\cdot\frac{1}{2}\cdot 1 = 1/4$

$f_z = \frac{1}{2}(x+yz)^{-1/2}\cdot y = \frac{1}{2}\cdot\frac{1}{2}\cdot 3 = 3/4$

$\nabla f\Big|_{(1,3,1)} = \left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right)$

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Panel 6

$f(x,y) = x^2y + 2xy^2 + 5y^3$ $\qquad f(tx,ty)$

$f(tx,ty) = (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 =$

$= t^3(x^2y + 2xy^2 + 5y^3) = t^3\cdot f(x,y)$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f(x,y)$ if f is homog. of degree n

$u = tx,\ v = ty$ $\quad = f(u,v)$ $\qquad$ set $t=1$

$\frac{\partial}{\partial t} f = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial t} =$

$\frac{\partial}{\partial t} f(tx,ty) = \frac{\partial}{\partial t} t^n f(x,y) = \frac{\partial f}{\partial u}\, x + \frac{\partial f}{\partial v}\, y$, $\quad n t^{n-1} f(x,y) = x\frac{\partial f}{\partial(tx)} + y\frac{\partial f}{\partial(ty)}$

6

Panel 7

Name: ______________

Quiz

① Consider the function $f(x,y) = x^2 + 3xy - y^2$. Find

a) $f_x = 2x+3y$

b) $\frac{\partial f}{\partial y} = 3x - 2y$

c) $\nabla f = \langle 2x+3y, 3x-2y \rangle$

② If $f(x,y) = xy + 3xy^2$ and $x = \sin(2t)$, $y = \cos(t)$. Find $\frac{\partial f}{\partial t}$ for $t = 0$: $f(t) = 8$

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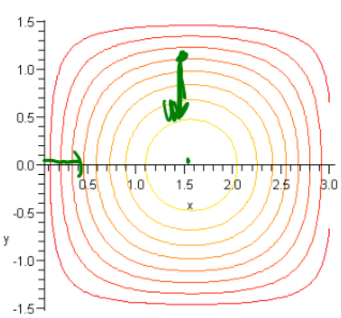
Panel 8

③ $f(x,y) = x^3 - 3xy + 4y^2$. Find directional derivative in the direction of $\langle \cos(\pi/6), \sin(\pi/6) \rangle$. $\rightarrow \| \langle \cos(\frac{\pi}{6}), \sin(\frac{\pi}{6}) \rangle \| = 1$

$\nabla f \cdot \vec{v} = \langle 3x^2 - 3y, -3x + 8y \rangle \cdot \langle \cos(\pi/6), \sin(\pi/6) \rangle = *$

④ Consider the contour plot below. Sketch the gradient



a) at $P(1.5, 1.0)$

b) at $P(0,0)$

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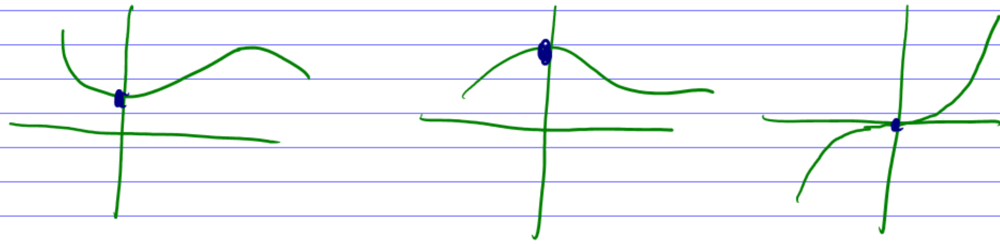
Panel 9

Review of Max / Min problems in $\mathbb{R}$



① $f'(x)=0$ or $f'(x)$ d.n.e. $\Rightarrow$ critical points

② If $f''(c)>0$ min

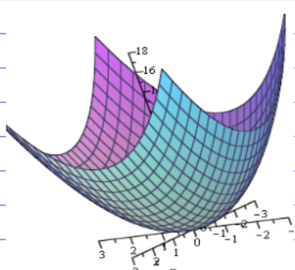
$f''(c)<0$. max

$f''(c)=0 \Rightarrow$ no info

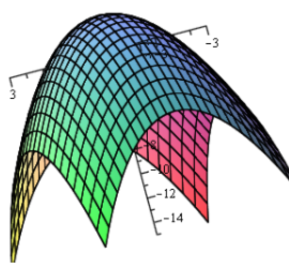
9

Panel 10

How about in $\mathbb{R}^2$

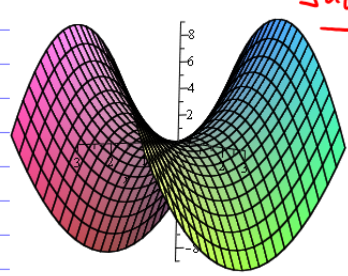


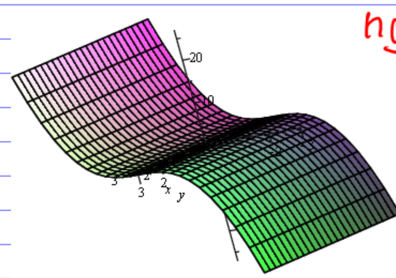
min



max

saddle





none

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Panel 11

Max / Min Problems

To find max/min of $z = f(x, y)$:

① Find $\nabla f = \langle f_x, f_y \rangle$

② Solve $\nabla f = 0$, i.e. $f_x = 0$, $f_y = 0$ simult.

③ Compute Hessian matrix $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ and $D = f_{xx} f_{yy} - (f_{xy})^2$ determinant

a) f has min if: $D > 0$, $f_{xx} > 0$

b) f has max if: $D > 0$, $f_{xx} < 0$

c) f has saddle if: $D < 0$

d) no information if: $D = 0$

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Panel 12

Ex: Find and classify the critical points for

$$f(x,y) = x^2 - 2xy + 3y^2 + 4x$$

$f_x = 2x - 2y + 4 = 0$

$f_y = -2x + 6y = 0$

$4y + 4 = 0$, $y = -1$, $x = -3$ critical point.

$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}$, $D = 2 \cdot 6 - (-2)(-2) > 0$

$D > 0$, $f_{xx} = 2 > 0$ $(-3, -1)$ is min

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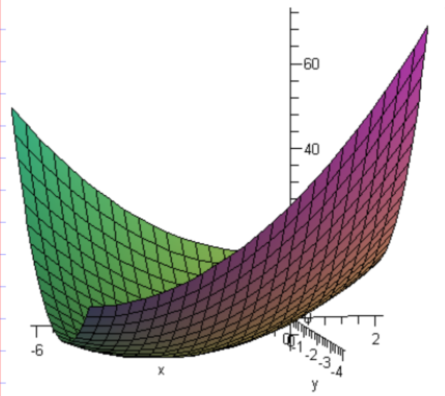
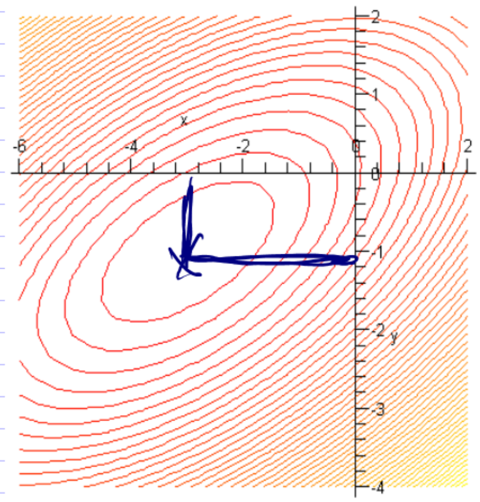
Panel 13

$f(x,y) = x^2 - 2xy + 3y^2 + 4x$





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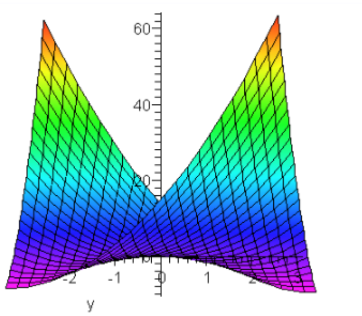
Panel 14

Suppose $f(x,y) = x^2 + 2y^2 + 4xy$. Find and classify all relative extrema, if any.



$f_x = 2x + 4y = 0$

$f_y = 4y + 4x = 0$

$x = y = 0$

$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix} = 8 - 16 < 0$

$D < 0 \Rightarrow$ saddle

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Panel 15

Find and classify critical points for $f(x,y) = 3x - x^3 - 2y^2$

$f_x = 3 - 3x^2 = 0 \qquad x = +1 \text{ or } -1$

$f_y = -4y = 0 \qquad y = 0$

two critical points: $(1,0)$ and $(-1,0)$

H, D $\qquad\qquad$ H, D

max $\qquad\qquad$ saddle

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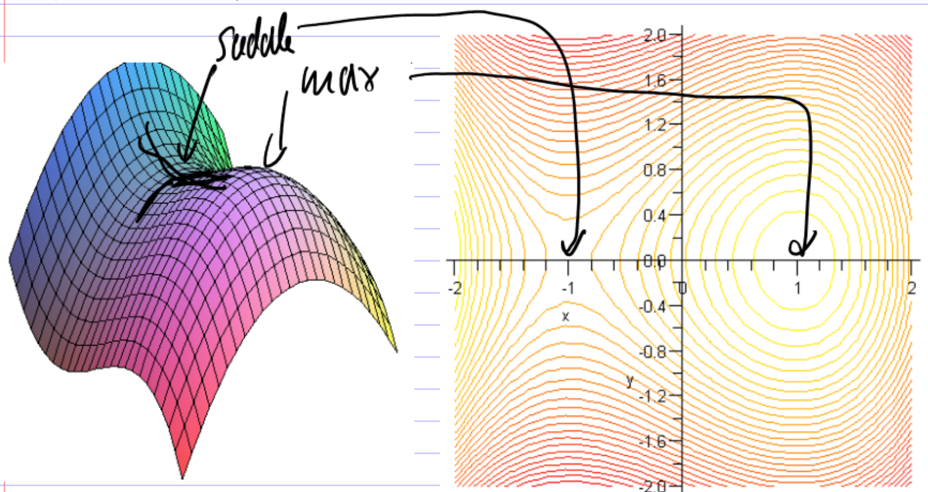
Panel 16

$f(x,y) = 3x - x^3 - 2y^2$



No class Wed, Oct. 19 (Sorry)

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