Panel 1

Last time

Partial derivatives: $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_z = \frac{\partial f}{\partial z}$

Higher-order partials: $f_{xyz} = \frac{\partial^3 f}{\partial x \partial y \partial z}$

Tangent planes: the plane that is tangent to the surface $z = f(x,y)$ at a point P

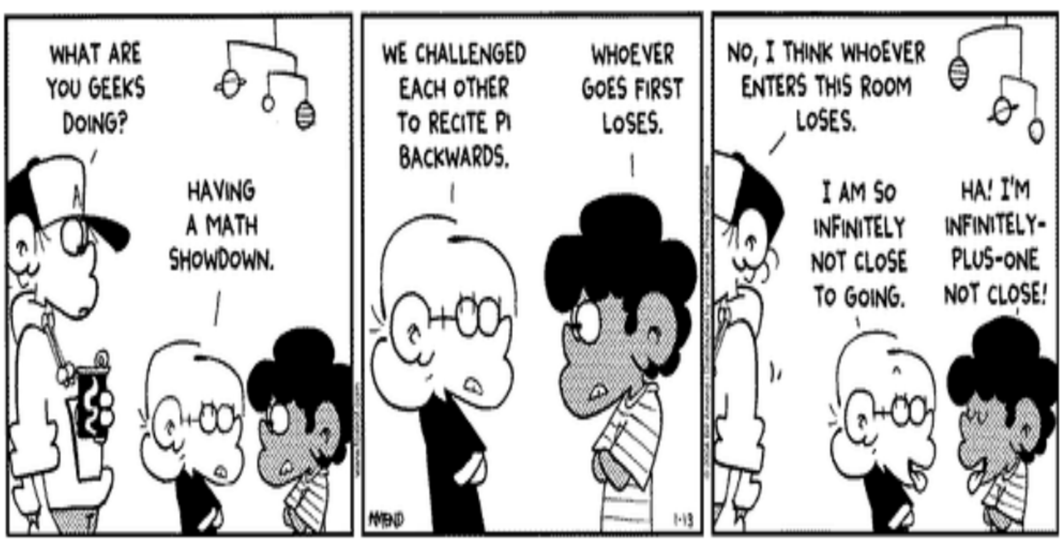
new def Harmonic functions

$\Delta f = f_{xx} + f_{yy} = 0$ (Laplacian)

Chain Rule: If $f(x,y)$ with $x = f(t)$, $y = f(t)$,

then $\frac{\partial f}{\partial t} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$

Panel 2



Panel 3

Ex: Let $f(x,y,z) = x y^2 \cos(xz)$. Find

$\frac{\partial^2 f}{\partial x \partial y}$: $f_x = y^2 \cos(xz) - xy^2 \sin(xz) \cdot z$

$f_y = 2\cos(xz) y - 2xzy \sin(xz)$

$\frac{\partial^3 f}{\partial y^2 \partial z}$: $f_y = 2y \cdot x\cos(xz)$

$f_{yy} = 2x\cos(xz)$

$f_{yyz} = -2x^2 \sin(xz)$

3

Panel 4

Ex: If $f(x,y) = \sin(x)\cosh(y)$ harmonic?

Recall: $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ $\left(\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})\right)$

$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ $\left(\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})\right)$

$\frac{\partial}{\partial x}\cosh(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$

$\frac{\partial}{\partial x}\sinh(x) = \cosh(x)$

f harmonic: $f_{xx} + f_{yy} = 0$

$f_x = \cos(x)\cosh(y)$, $f_{xx} = -\sin(x)\cosh(y)$

$f_y = \sin(x)\sinh(y)$, $f_{yy} = \sin(x)\cosh(y)$

$\Rightarrow$ $f_{xx} + f_{yy} = 0$ ✓

YES

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Panel 5

Equation of tangent plane to $f(x,y)$ at (x_0,y_0) is:

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

Ex: $f(x,y) = x^2 - 3xy + y^2$ when $x=1$ and $y=1$

$$f_x = 2x - 3y, \quad f_x(1,1) = -1$$
$$f_y = -3x + 2y, \quad f_y(1,1) = -1$$
$$z_0 = f(1,1) = -1$$

tangent plane: $z = -(x-1) - (y-1) - 1$

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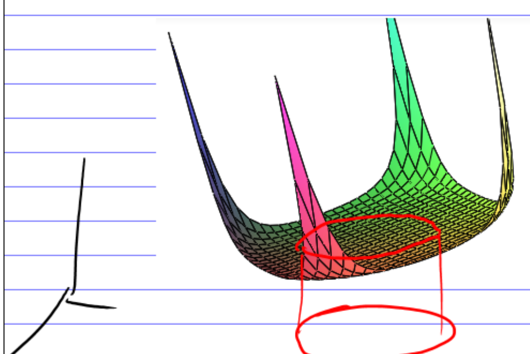
Panel 6

The Chain Rule

$z = f(x,y)$ where $x = f(t)$ and $y = g(t)$

$$\Rightarrow \frac{\partial z}{\partial t} = \frac{\partial f}{\partial t} = \frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Ex: $f(x,y) = e^{x^2+y^2}$, $x = \cos(t)$, $y = \sin(t)$ $\Rightarrow f(t) = e^{\cos^2(t)+\sin^2(t)} = e^1$



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} =$$
$$= 2x e^{x^2+y^2}(-\sin(t)) + 2y e^{x^2+y^2}\cos(t) =$$
$$= 2e^{x^2+y^2}(-\cos(t)\sin(t) + \sin(t)\cos(t)) =$$
$$= 0$$

6

Panel 7

Ex: Suppose $x^3 + y^3 = 6xy$ defines $y = y(x)$ implicitly

$f(x,y) = x^3 + y^3 - 6xy = 0$ Consider $\boxed{y = y(x)}$ $(\cancel{x = x(y)})$

Want: $y'(x)$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} \cdot \underbrace{\cancel{\frac{\partial x}{\partial x}}}_{1} + \frac{\partial f}{\partial y} \cdot \underbrace{\cancel{\frac{\partial y}{\partial x}}}_{y'} = 0 \Rightarrow (3x^2 - 6y) 1 + (3y^2 - 6x) y' = 0$$

$$y' = -\frac{3x^2 - 6y}{3y^2 - 6x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} \cdot \underbrace{\cancel{\frac{\partial x}{\partial y}}}_{x'} + \frac{\partial f}{\partial y} \cdot \underbrace{\cancel{\frac{\partial y}{\partial y}}}_{1} \Rightarrow x' = \ldots$$

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Panel 8

Thm: If $f(x,y)$ is a function such that all 2nd-order partials exist and are continuous almost always

then $f_{xy} = f_{yx}$

8

Panel 9

Directional Derivatives

$f_x = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$ deriv in $\langle 1,0 \rangle$ direction (x)

$f_y = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$ deriv. in $\langle 0,1 \rangle$ direction (y)

Def: Directional derivative in direction $\vec{u} = \langle a,b \rangle$, where u is a unit vector, is:

$$D_{\vec{u}}(f) = \lim_{h \to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h}$$

$D_{\langle 1,0 \rangle}(f) = f_x$, $\quad D_{\langle 0,1 \rangle}(f) = f_y$

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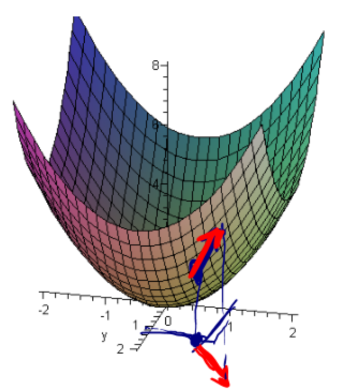
Panel 10

Ex: $f(x,y) = x^2 + y^2$. Find directional derivative of f in the direction of $\vec{u} = \frac{1}{\sqrt{2}}\langle 1,1 \rangle$ at $(1,1)$:



$$D_u(f) = \lim_{h \to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+\frac{h}{\sqrt{2}})^2 + (y+\frac{h}{\sqrt{2}})^2 - (x^2+y^2)}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2x\frac{h}{\sqrt{2}} + \frac{h^2}{2} + \cancel{y^2} + 2y\frac{h}{\sqrt{2}} + \frac{h^2}{2} - \cancel{x^2} - \cancel{y^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(\sqrt{2}x + \frac{h}{2} + \sqrt{2}y + \frac{h}{2})}{\cancel{h}} = \sqrt{2}(x+y) \quad \text{at } (1,1): D_u(f) = \underline{\sqrt{2}\,2}$$

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Panel 11

Thm: Suppose $\vec{u} = \langle a, b \rangle$ is a unit vector. Then

$$D_{\vec{u}} f(x,y) = \langle f_x, f_y \rangle \cdot \vec{u} = \langle f_x, f_y \rangle \cdot \langle a, b \rangle = a f_x + b f_y$$

Ex: $f(x,y) = x^2 + y^2$, $\vec{u} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$. Find $D_{\vec{u}} f(x,y)$ at $(1,1)$

$$D_u(f) = \langle f_x, f_y \rangle \cdot \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \rangle =$$

$$= \langle 2x, 2y \rangle \cdot \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \rangle = \frac{2}{\sqrt{2}} x + \frac{2}{\sqrt{2}} y = \sqrt{2}(x+y)$$

same

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Panel 12

Ex: $f(x,y) = x^3 - 3xy + 4y^2$. Find directional derivative in the direction of $\langle \cos(\pi/6), \sin(\pi/6) \rangle$

$$D_u(f) = \langle f_x, f_y \rangle \cdot \langle \cos(\pi/6), \sin(\pi/6) \rangle$$

$$= \langle 3x^2 - 3y, -3x + 8y \rangle \cdot \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle$$

$$= (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y) \cdot \frac{1}{2} \qquad (\#)$$

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Panel 13

Def: If $f(x,y)$ is a function of 2 variables, then the <u>gradient of f</u> is:

$$\nabla f := \langle f_x, f_y \rangle \qquad (\text{or } \nabla f = \langle f_x, f_y, f_z \rangle)$$

Gradient

Note: $D_{\vec{u}} f = \langle f_x, f_y \rangle \cdot \vec{u} = \nabla f \cdot \vec{u}$ $\qquad \cancel{\Delta f = f_{xx} + f_{yy}}$

Ex: $f(x,y) = xy\sin(x)$.

$$\nabla f = \langle y\sin(x) + xy\cos(x),\ x\sin(x) \rangle$$

$$f(x,y,z) = x\cos(y^2+z^2)$$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle \cos(y^2+z^2), -x2y\sin(y^2+z^2), -x2z\sin(y^2+z^2) \rangle$$

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Panel 14

Recall: $D_{\vec{u}} f = \nabla f \cdot u = \cos(\theta)\,\|\nabla f\|\,\|u\|$

$$= \cos(\theta)\,\|\nabla f\|$$

Theorem: The max. value of $D_{\vec{u}}(f)$ is $\|\nabla f\|$ and is attained if $\vec{u}$ points in direction of ∇f (because $\theta = 0$, $\cos(\theta) = 1$)

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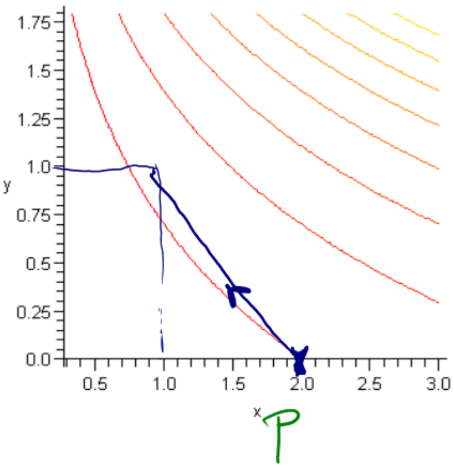
Panel 15

Ex: $f(x,y) = xe^y$. Find rate of change at $P(2,0)$ in the direction from P to $Q(1,1)$.



unit vector

$$D_{\vec{u}}(f) = \langle f_x, f_y \rangle \cdot \frac{1}{\sqrt{2}}\langle -1,1 \rangle$$

$$f_x = e^y \Big|_{(2,0)} = 1$$

$$f_y = xe^y \Big|_{(2,0)} = 2$$

$$D_u(f)\Big|_{(2,0)} = \langle 1,2 \rangle \cdot \frac{1}{\sqrt{2}}\langle -1,1 \rangle =$$

$$= \frac{1}{\sqrt{2}}(-1+2) = 1/\sqrt{2} = 0.7$$

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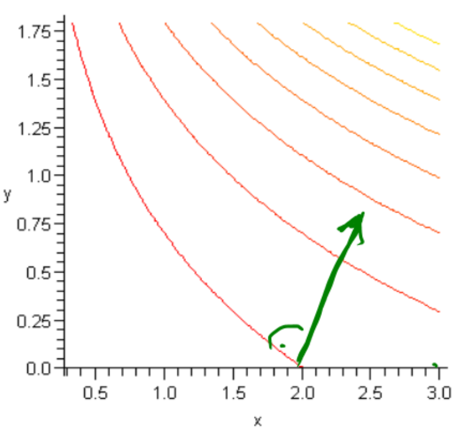
Panel 16

Ex: $f(x,y) = xe^y$. You are standing at $P(2,0)$. In which direction does f change most rapidly, and what is that rate of change?



Dir of most rapid change is $\nabla f = \langle 1,2 \rangle$

Max. rate of change

$$\|\nabla f\| = \sqrt{5}$$

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