Panel 1

Last time: Review of contour plots and surfaces. $z = f(x,y)$

limits: if $\|(x,y)-(x_0,y_0)\| < \delta$ then $|f(x,y) - L| < \varepsilon$

continuity: $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$ (y is fixed)

$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$ (x is fixed)

$\frac{\partial^2 f}{\partial x \partial y} = (f_x)_y$

1

Panel 2

Ex: Let $f(x,y,z) = \underbrace{xy}\, z\, \underbrace{\sin(z)}$. Find

$\frac{\partial^2 f}{\partial x \partial y}$:

$f_x = yz\sin(z)$

$f_{xy} = (f_x)_y = \underline{\underline{z\sin(z)}}$

$\frac{\partial^3 f}{\partial z \partial y^2}$: $f_z = xy\sin(z) + xyz\cos(z)$

$f_{zy} = x\sin(z) + xz\cos(z)$

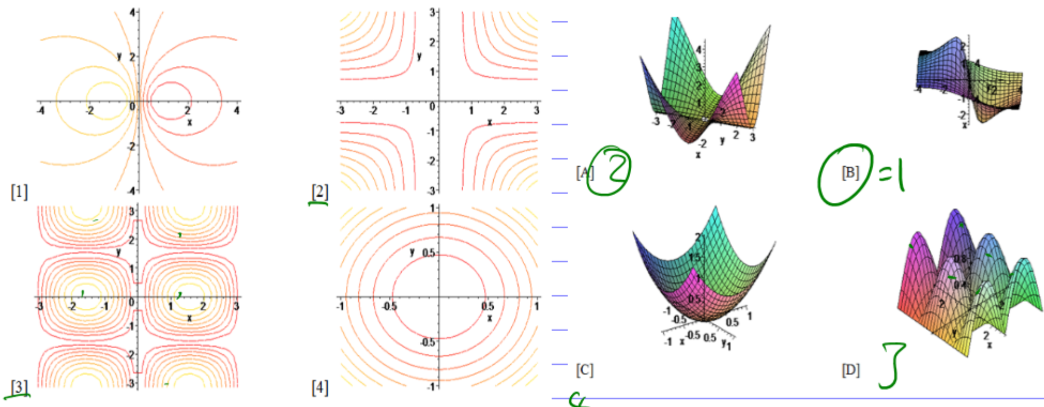
$f_{zyy} = 0$

2

Panel 3

Quiz #5 Name: ____________

① Match the surfaces on the right with the contour plots on the left.



3

Panel 4

#2) Find the limit if possible. Justify your argument.

$$\lim_{(x,y)\to(0,0)} \frac{3xy^4}{4x^2+5y^8}$$

$x=0, y\to 0$: lim $= 0$

$y=0, x\to 0$: lim $= 0$

$x=y \to 0$: lim $= 0$

$x=y^4 \to 0$

$$\frac{3x^5}{4x^4+5x^8} = \frac{3x^{\cancel{5}\,3}}{\cancel{x^4}(4+5x^4)} = 0$$

$$\cancel{\frac{3y^8}{4y^8+5y^8}} = \frac{1}{3}$$

#3) Find the indicated partial derivatives

a) $f(x,y) = xy + x^2y^3$. $f_x = y + 2xy^3$

$f_y(x,y) = x + 3x^2y^2$

b) $f(x,y,z) = yz\sin(xz)$

$\frac{\partial f}{\partial z}$ = did this

4

Panel 5

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2-y^2} = \lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2-y^2)}(x^2+y^2)}{\cancel{x^2-y^2}} = 0$$

5

Panel 6

Partial derivatives frequently occur in Physics to describe laws of nature as PDEs (partial differential equations). For example: the Laplace PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or} \quad u_{xx} + u_{yy} = 0$$

is important in heat conduction and fluid flow.

Ex: Show that $f(x,y) = e^x \sin(y)$ satisfies the above PDE

$f_x = e^x \sin(y)$ $\qquad$ $f_{xx} = e^x \sin(y)$

$f_y = e^x \cos(y)$ $\qquad$ $f_{yy} = -e^x \sin(y)$

$f_{xx} + f_{yy} = 0$

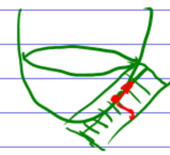
6

Panel 7

f_x = slope of tangent in x-dir.

f_y = slope of tangent in y-dir.

Find the tangent plane to $z = f(x,y)$



Can not take cross product of f_x by f_y (not vectors)

Plane: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ WOLOG: $c=1$

$$\frac{a}{c}(x-x_0)+\frac{b}{c}(y-y_0)+(z-z_0)=0$$

$$\Rightarrow \quad z = -\frac{a}{c}(x-x_0) - \frac{b}{c}(y-y_0)+z_0$$

$f(x,y) = z = A(x-x_0)+B(y-y_0)+z_0$ is tangent plane function.

$$f_x = A = f_x, \quad f_y = B = f_y$$

7

Panel 8

Equation of tangent plane to $f(x,y)$ at (x_0,y_0) is:

$$z = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+z_0$$

Ex: $f(x,y) = 2x^2+y^2$. Find tangent plane at $P(1,1,3)$ ✓

$f_x = 4x$ $f_y = 2y$

$f_x(1,1) = 4$, $f_y(1,1) = 2$

$$z = 4(x-1)+2(y-1)+3$$

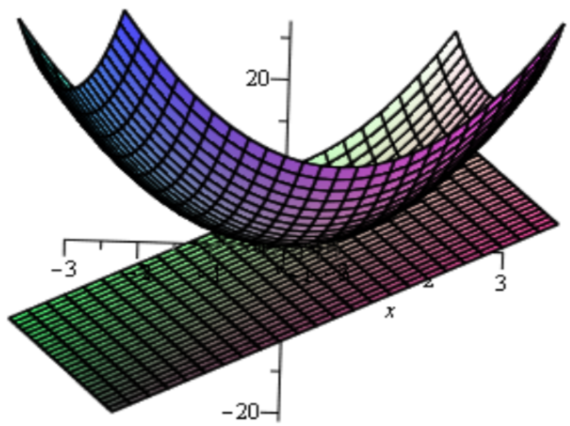
8

Panel 9

$f(x,y) = 2x^2 + y^2$ Tangent plane at $P(1,1,3)$ is:

$z = 4(x-1) + 2(y-1) + 3$

$plot3d(\{2 \cdot x^2 + y^2, 4 \cdot (x-1) + 2 \cdot (y-1) + 3\}, x = -3..3, y = -3..3)$



9

Panel 10

The Chain Rule

$f: \mathbb{R} \to \mathbb{R},\ g: \mathbb{R} \to \mathbb{R} \Rightarrow \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

Chain Rule in $\mathbb{R}^2$:

$z = f(x,y),\ x = g(t),\ y = h(t) \Rightarrow f(g(t), h(t))$

$$\Rightarrow \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$z = f(x,y),\ x = g(s,t),\ y = h(s,t) \Rightarrow f(g(s,t), h(s,t))$

$$\Rightarrow \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

10

Panel 11

Ex: $f(x,y) = x^2 y + 3xy^4$, $x = \sin(t)$, $y = \cos(t)$. circle Find $\frac{\partial f}{\partial t}$ at $t=0$

without chain rules: $x = \sin(t)$, $y = \cos(t)$, $z = x^2 y + 3xy^4$

$\Rightarrow z = \sin^2(t)\cos(t) + 3\sin(t)\cos^4(t)$ at $t=0$: 3

$$\frac{\partial z}{\partial t} = \frac{dz}{dt} = 2\sin(t)\cos^2(t) - \sin^3(t) + 3\cos^5(t) - 12\sin^2(t)\cos^3(t)$$

with chain rules $x = \sin(t)$, $x(0) = 0$; $y(t) = \cos(t)$, $y(0) = 1$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2xy + 3y^4)\cos(t) - (x^2 + 12xy^3)\sin(t)$$

at $t=0$: 3

11

Panel 12

Chain Rule is useful for Implicit Differentiation

Suppose $F(x,y) = 0$ is a function defining x and y implicitly, or more precisely, defines $y = y(x)$ implicitly.

$F(x,y) = 0$ but $y = y(x)$.

$$\frac{\partial}{\partial x} F(x,y) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0$$

$$F_x \cdot 1 + F_y \cdot y' = 0 \quad \Rightarrow y' = -F_x / F_y$$

12

Panel 13

Ex: Find y' if $x^3 + y^3 = 6xy$

$F = x^3 + y^3 - 6xy = 0 \quad | \frac{\partial}{\partial x}$

$\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} = (3x^2 - 6y) \cdot 1 + (3y^2 - 6x) y' = 0$

$y' = -\frac{3x^2 - 6y}{3y^2 - 6x}$

Ex: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$

on Friday

13