Panel 1

Last time

$f: \mathbb{R}^2 \to \mathbb{R}$ e.g. $f(x,y) = x^2+y^2$ (Graph is 3D)

$f: \mathbb{R}^3 \to \mathbb{R}$ e.g. $f(x,y,z) = x^2+y^2+z^2$ (Graph is 4D)

Graphs + Slices

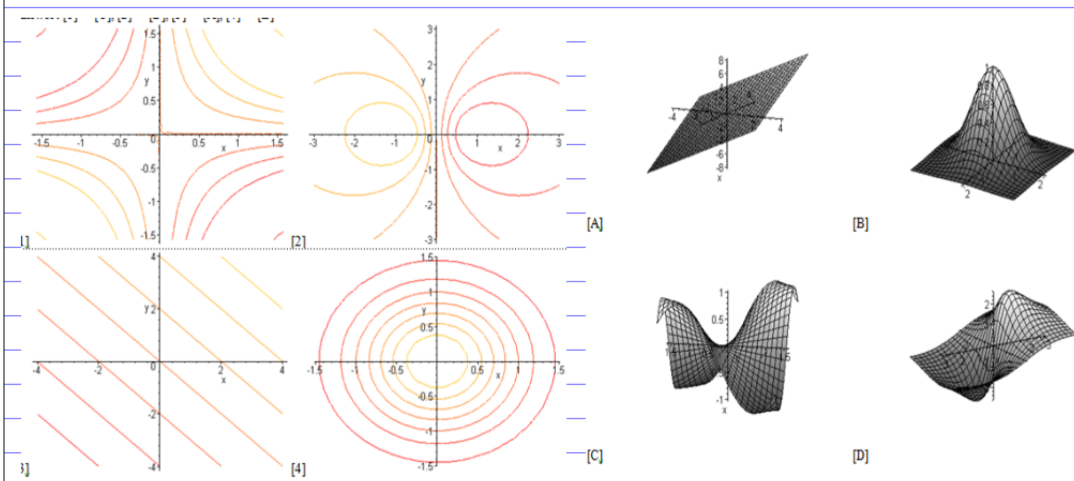
Level Curves $f(x,y) = \text{const}$

Contour Plots →

Limits

1

Panel 2



Match Contour plots to Graphs

2

Panel 3

Limits: $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

Def: Given any $\varepsilon > 0$ there is a $\delta > 0$ such that

whenever $\|(x,y) - (x_0,y_0)\| < \delta$ then $|f(x,y) - L| < \varepsilon$

How to really find limits

1.) Substitute point and see what you get.

2.) $x = x_0$, $y \to y_0$

$y = y_0$, $x \to x_0$

$x = y_0$ $x \to x_0$

$x = y^2$ or $y = x^2$, $x \to x_0$

See if any are different. If so, limit d.n.e.

3.) Try to prove the limit is L.

3

Panel 4

Ex: $\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2} = \frac{1}{2}$

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = $ D.N.E.

$x = 0$: $y \to 0$: $\lim = 0$

$y = 0$: $x \to 0$: $\lim = 0$

$x = y$, $x \to 0$: $\lim$ $\frac{1}{2}$

lim

4

Panel 5

Ex: Find $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ ~~if it exists~~ D.N.E.

$x=0$: $\lim = 0$

$y=0$: $\lim = 0$

$x=y$: $\lim = \frac{x^3}{x^2+x^4} = \frac{x^{\cancel{3}}}{\cancel{x^2}(1+x^2)} = \frac{x}{1+x^2} \to 0$

$x=y^2$: $\lim \frac{y^4}{y^4+y^4} = \frac{1}{2}$

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Panel 6

Ex: $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

All approaches get 0 as limit.

6

Panel 7

Prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

Take $\varepsilon > 0$ find $\delta > 0$ s.t.

$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| < \varepsilon$ if $\|(x,y)-(0,0)\| = \delta$

if $\sqrt{x^2+y^2} < \delta$

Note: $x^2 < x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2} < 1$

$\Rightarrow \left| \frac{3x^2y}{x^2+y^2} \right| = \frac{3|y|\,x^2}{x^2+y^2} < 3|y| = 3\sqrt{y^2} < 3\sqrt{x^2+y^2}$

Take $\varepsilon > 0$, I pick $\delta = \varepsilon/3$. Then $|f(x,y)| < 3\delta = 3\frac{\varepsilon}{3} = \varepsilon$

$\|(x,y)\| < \delta$

Panel 8

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ d.n.e.

$x = 0$, then $\lim = 0$

$y = 0$: then $\lim = 1$

Ex: $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ d.n.e.

$x = 0$: $\lim = 0$

$y = x^2$: $\lim \frac{2x^4}{x^4+x^4} = 1$

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$

$x=0$: $\lim = 0$, $y = 0$: $\lim \frac{x^3}{x^2} = 0$, $x = y$: $\lim = 0$, $x = y^2$, $y = x^2$: 0

Panel 9

$\lim \frac{x^3}{x^2+y^2} = 0$: Given $\varepsilon > 0$: pick $\delta > 0$ s.t.

$\left|\frac{x^3}{x^2+y^2}\right| < \varepsilon$ as long as $\|(x,y)\| = \sqrt{x^2+y^2} < \delta$

$$\left(\left|\frac{x \cdot x^2}{x^2+y^2}\right| = |x| \frac{x^2}{x^2+y^2} \le |x| = \sqrt{x^2} \le \boxed{\sqrt{x^2+y^2}} < \delta\right)$$

Proof: Take $\varepsilon > 0$, pick $\boxed{\delta = \varepsilon}$. Then if

$\sqrt{x^2+y^2} < \delta$ then $|f(x,y)| \le \sqrt{x^2+y^2} < \delta = \varepsilon$

$|f(x,y)| < \varepsilon$

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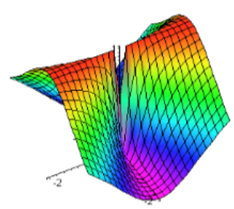
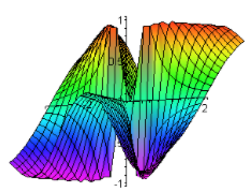
Panel 10

$f(x,y) = \frac{x^2}{x^2+y^2}$

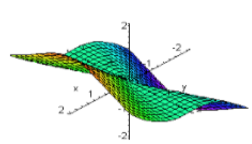


$f(x,y) = \frac{2x^2 y}{x^4+y^2}$



no limit at (0,0)

$f(x,y) = \frac{x^3}{x^2+y^2}$



limit exists

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Panel 11

Continuity

As usual, continuity is just a restatement of limits

Def. $f(x,y)$ is continuous at (x_0,y_0) if

$$\lim_{(x_0,y_0)\to(x,y)} f(x,y) = f(x_0,y_0)$$

Ex:

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ \cancel{2}\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Is f cont. at $(1,1)$: $\lim_{(x,y)\to(1,1)} f(x,y) = 3/2 = f(1,1)$

Is f cont. at $(0,0)$: $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ not cont. is cont.

11

Panel 12

Derivatives:

If $f(x,y)$ is a function of 2 variables, define

$$\frac{\partial f}{\partial x} = f_x = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$ partial deriv. with respect to x

$$\frac{\partial f}{\partial y} = f_y = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$ to y

Recall: $f: \mathbb{R} \to \mathbb{R}$

$$f'(x) = \frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

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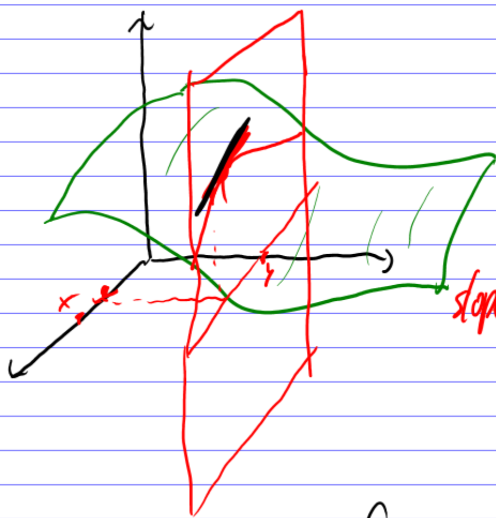
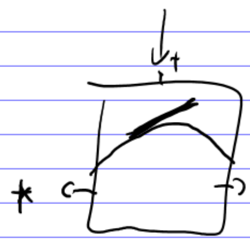
Panel 13



f_x: $\lim\limits_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$

slope of tangent to $f(x,y)$, for y fixed, i.e. parallel to (x,z)-plane



f_y: is slope of tangent to $f(x,y)$ for x fixed, i.e. parallel to (y,z)-plane

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Panel 14

Ex: Find f_x if $f(x,y) = x^2y + y^2$

$$f_x = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0} \frac{(x+h)^2y + y^2 - x^2y - y^2}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2y} + 2xhy + h^2y \cancel{+y^2} - \cancel{x^2y} - \cancel{y^2}}{h}$$

$$= \lim_{h\to 0} \frac{2xyh + h^2y}{h} = \lim_{h\to 0} 2xy + hy = 2xy$$

$$f_y = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = x^2 + 2y$$

HW

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Panel 15

How to Really do partial derivatives

f_x: think of y as const., take the usual deriv. for x

f_y: think of x as const., take the usual deriv. for y

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Panel 16

Ex: $f(x,y) = x^3 + x^2 y^3 - 2y^2$. Find

$f_x(2,1) = 3(2)^2 + 2 \cdot 2 \cdot 1 = \underline{\underline{16}}$

$f_x(x,y) = 3x^2 + 2xy^3$

$f_y(2,1) = 3(2)^2 1^2 - 4 \cdot 1 = \underline{\underline{8}}$

$f_y(x,y) = 3x^2y^2 - 4y$

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Panel 17

3D Example: $f(x,y,z) = \underbrace{xz}_{}\,\underbrace{e^{x^2+y^2}}_{c}$. Find

$$f_x(x,y,z) = \frac{\partial f}{\partial x} = z \cdot e^{x^2+y^2} + xz\, e^{x^2+y^2}(2x)$$

$$= e^{x^2+y^2}(z + 2x^2 z)$$

$$f_y(x,y,z) = \frac{\partial f}{\partial y} = xz\, e^{x^2+y^2}(2y) = 2xyz\, e^{x^2+y^2}$$

$$f_z(x,y,z) = \frac{\partial f}{\partial z} = e^{x^2+y^2} \cdot x = x e^{x^2+y^2}$$

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Panel 18

Of course we can also take higher-order partial derivatives. Let $f(x,y)$ be a function

1st order: f_x f_y

2nd order: f_{xx} f_{xy} f_{yx} f_{yy}

3rd order: f_{xxx} f_{xxy} f_{xyx} f_{xyy} f_{yxx}, f_{yxy} f_{yyx} f_{yyy}

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Panel 19

Ex: $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x(x,y) = 3x^2 + 2xy^3$

$f_y(x,y) = x^2 3y^2 - 4y$

$f_{xx}(x,y) = 6x + 2y^3$

$$\left.\begin{array}{l} f_{xy}(x,y) = 6xy^2 \\ f_{yx}(x,y) = 6xy^2 \end{array}\right\} =$$

$f_{yy}(x,y) = 6x^2y - 4$

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Panel 20

Notation:

$$f_x = \frac{\partial f}{\partial x}$$

$$f_y = \frac{\partial f}{\partial y}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

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Panel 21

Ex: Let $f(x,y,z) = \underbrace{xyz}\,\underbrace{\sin(z)}$. Find

$\frac{\partial^2 f}{\partial x \partial y}$: $f_x = yz\sin(z)$

$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = (f_x)_y = z\sin(z)$

$\frac{\partial^3 f}{\partial z \partial y^2}$: $f_z = xy\sin(z) + xyz\cos(z)$

$f_{zy} = x\sin(z) + xz\cos(z)$

$\partial_{zyy} = 0$

Quiz on Friday

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