Panel 1

Functions of Several Variables

Know: $f: \mathbb{R} \to \mathbb{R}$

e.g.

e.g.

e.g.

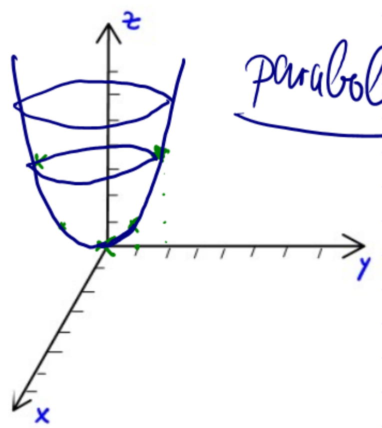
1

Panel 2

Def: A function of 2 variables is a rule that assigns to every pair (x,y) in a set $D \subset \mathbb{R}^2$ exactly one number $z = f(x,y)$

Ex: $f(x,y) = x^2 + y^2$

x	y	$z = f(x,y)$
0	0	0
0	1	1
1	0	1
1	1	2
0	2	4
0	-2	4



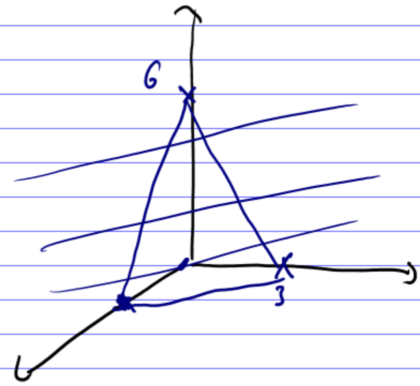
paraboloid

2

Panel 3

Ex: $f(x,y) = 6 - 3x - 2y$

$z = 6 - 3x - 2y$ or $3x + 2y + z = 6$



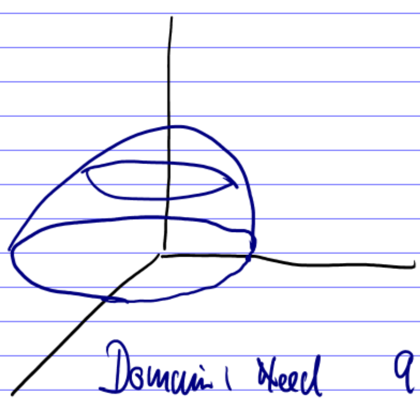
3

Panel 4

Ex: $f(x,y) = \sqrt{9 - x^2 - y^2} = z$

$9 - x^2 - y^2 = z^2$ so $x^2 + y^2 + z^2 = 9$

full sphere



upper half sphere

(note: $f: \mathbb{R}^2 \to \mathbb{R}$ have to pass vertical line test)

Domain: Need $9 - x^2 - y^2 \geq 0$, $9 - x^2 - y^2 \geq 0$

$9 \geq x^2 + y^2$

Domain: inside circle, radius 3

4

Panel 5

$f:\mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^4 + y^3 + z^2$

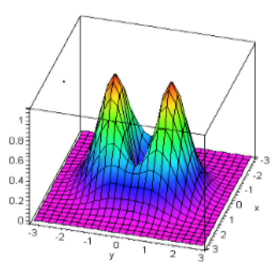
Graphs would be 4D - can't "see" this

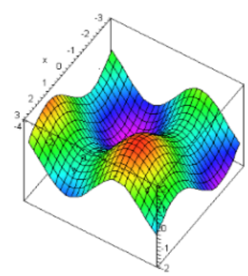
5

Panel 6

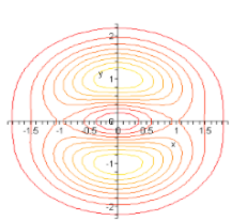
$f(x,y) = (x^2+3y^2)\, e^{-x^2-y^2}$

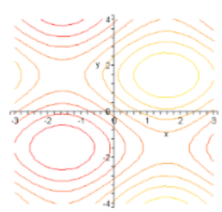
$f(x,y) = \sin(x) + \sin(y)$





<u>Level curves</u>: curves in xy-plane where $f(x,y)$ is <u>constant</u>





aka topographical maps

6

Panel 7

Of course I used Maple to generate these plots

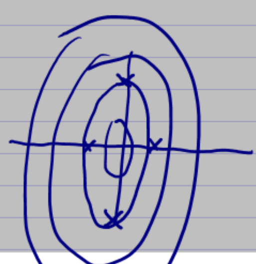
```
> plot3d((x^2+3*y^2)*exp(-x^2-y^2),x=-3..3,y=-4..4);
> plot3d(sin(x)+sin(y),x=-3..3,y=-4..4);
> with(plots);
> contourplot((x^2+3*y^2)*exp(-x^2-y^2),x=-3..3,y=-4..4);
> contourplot(sin(x)+sin(y),x=-3..3,y=-4..4);
>
```

7

Panel 8

Ex: Level curves of $h(x,y) = 4x^2+y^2$

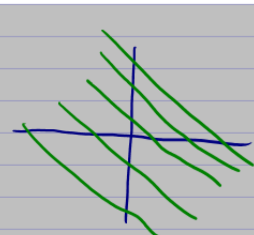
Level curves: $4x^2+y^2 = c$ $(=1)$



Ex: Level curves for $f(x,y) = 6-3x-2y$

$6-3x-2y = c$

$6-c = 3x+2y$



3D is a plane

8

Panel 9

Limits: The limit of $f(x,y)$ as (x,y) approaches (x_0,y_0) is L is written as

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

Given $\varepsilon > 0$ there is $\delta > 0$ such that:

if $\|(x,y) - (x_0,y_0)\| < \delta$, i.e. $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

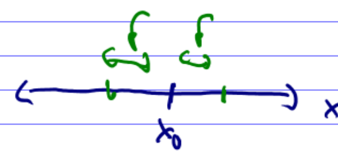
then $|f(x,y) - L| < \varepsilon$

Note: (x,y) can approach (x_0,y_0) from any direction

9

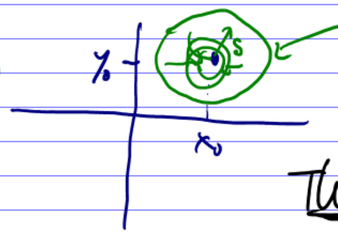
Panel 10

In $\mathbb{R}$: $\lim_{x\to x_0} f(x)$



In $\mathbb{R}^2$: $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

inf. ways to approach (x_0,y_0)



$(x-x_0)^2+(y-y_0)^2 \le \delta$

Thus: can not use right/left limits to show that a limit exists.

But: If you find 2 ways to approach (x_0,y_0) with different limits, then limit d.n.e.!

10

Panel 11

Hints for finding limits in $\mathbb{R}^2$:

→ if C_1 is a path to (x_0, y_0) and $f(x,y) \to L_1$ on C_1

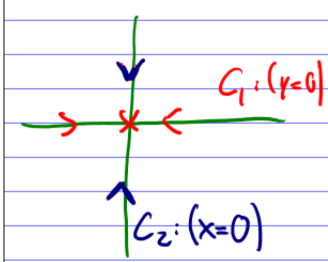
→ if C_2 is a path to (x_0, y_0) and $f(x,y) \to L_2$ on C_2

If $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Ex: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist

check $\lim\limits_{(x,0)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$

no limit



check $\lim\limits_{(0,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = \lim\limits_{y\to 0} \dfrac{-y^2}{y^2} = -1$

11

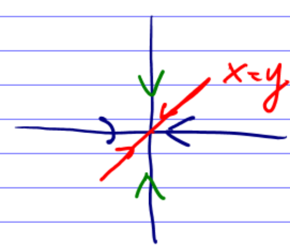
Panel 12

Ex: $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy}{x^2+y^2} = 1/2$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$: D.N.E

$x=0, y\to 0$: $\lim\limits_{y\to 0} \dfrac{0}{y^2} = \lim\limits_{y\to 0} 0$

$x\to 0, y=0$: $\lim\limits_{x\to 0} \dfrac{0}{x^2} = 0$



$x=y, \to 0$: $\lim\limits_{t\to 0} \dfrac{t^2}{2t^2} = \dfrac{1}{2} =$

$\ \ \ \ \parallel$

$\ \ \ \ t$

12

Panel 13

Ex: Find $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ ~~if it~~ does not exists

$x=0, y\to 0$: 0

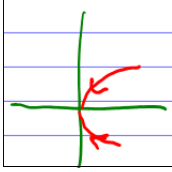
$y=0, x\to 0$: 0

l'Hospital

$x=y\to 0$: $\lim_{x\to 0} \frac{x^3}{x^2+x^4} = \lim_{x\to 0} \frac{3x^2}{2x+4x^3} = \lim_{x\to 0} \frac{6x}{2+12x^2} = 0$

Let $x=y^2\to 0$: $\lim_{y\to 0} \frac{y^4}{y^4+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$.

different



13

Panel 14

14

Panel 15

Ex: $\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2+y^2}$

go to (0,0) in a spiral

$(t\cos(t),\ t\sin(t))$

$x = t\cos(t)$, $y = t\sin(t)$

$x=0, y\to 0$: 0

$x\to 0, y=0$: 0

$\lim_{t\to 0} \frac{3t^2\cos^2(t)\, t\sin(t)}{t^2(\underbrace{\cos^2(t)+\sin^2(t)}_{1})} = 0$

$x=y\to 0$: $\lim_{t\to 0} \frac{3x^3}{2x^2} = \lim_{x\to 0} \frac{3}{2}x = 0$

give up - maybe limit is zero

$x=y^2\to 0$: $\lim_{y\to 0} \frac{3y^5}{y^4+y^2} = 0$

$y=x^2\to 0$: $\lim_{x\to 0} \frac{3x^4}{x^2+x^4} = \lim \frac{3x^{\cancel{4}\,2}}{\cancel{x^2}(1+x^2)} = 0$

15

Panel 16

Prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2+y^2} = 0$

Take any $\varepsilon > 0$, I need to find this $\delta > 0$

s.t.

next time

16