Panel 1

Summary

$\vec{r}(t)$ = space curve

$\vec{r}'(t)$ = tangent

$\vec{T}(t) = \frac{r'(t)}{\|r'(t)\|}$ unit tangent

$\vec{N}(t) = \frac{T'(t)}{\|T'(t)\|}$ principal; unit normal

$\vec{B}(t) = T(t) \times N(t)$ unit normal

$\kappa(t) = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$ curvature

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Panel 2

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find tangent, unit normal and binormal vectors at $t=0$

$r(0) = (1, 0, 0)$

$T(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$

$N(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

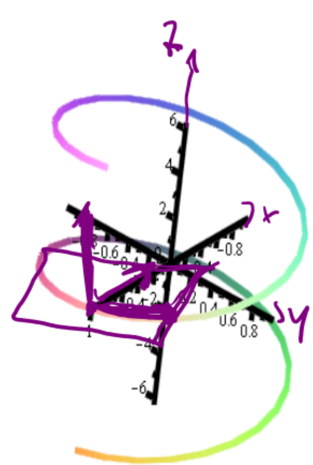
$B(t) = \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle$

$t=0$:

$\Rightarrow T(0) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow N(0) = \langle -1, 0, 0 \rangle$

$\Rightarrow B(0) = \left\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$



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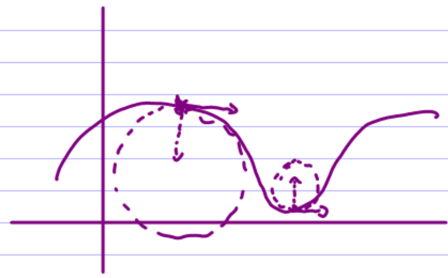
Panel 3

Def: The plane determined by T and N is called osculating plane or supporting plane.

Latin: osculum = kiss

Def: The circle in the osculating plane with radius $r = 1/\kappa$ is called the osculating circle



larger curvature = small radius.

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Panel 4

Ex: Find the osculating plane of

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle \quad \text{at} \quad P(0, 1, \pi/2).$$

We know

$$T(t) = 1/\sqrt{2}\,(-\sin(t), \cos(t), 1)$$

$$N(t) = (-\cos(t), -\sin(t), 0)$$

$$B(t) = \frac{1}{\sqrt{2}}(\sin(t), -\cos(t), 1)$$

$$B\left(\frac{\pi}{2}\right) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \quad \text{or} \quad \langle 1, 0, 1 \rangle$$

$$x + b = 0$$

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Panel 5

Ex: Find the osculating circle to $y = x^2$ at $(0,0)$.

Clearly for a 2D graph the osc. plane is $z=0$

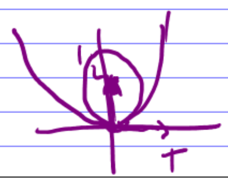
$\vec r(t) = \langle t, t^2, 0\rangle$ r

$r'(t) = \langle 1, 2t, 0\rangle$ $r'(0) = \langle 1, 0, 0\rangle$

$r''(t) = \langle 0, 2, 0\rangle$ $r''(0) = \langle 0, 2, 0\rangle$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2\rangle$$

$\kappa = 2/\|1\|^3 = 2$ $\Rightarrow$ radius is $1/2$



$(x)^2 + (y - 1/2)^2 = 1/4$

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Panel 6

Motion in Space

Suppose $\vec r(t)$ represents the motion or path of a particle through space over time:

$\vec r(t)$ = motion in space

$\vec r'(t)$ = velocity, $r'(t) = \vec v(t)$

$= \|\vec r'(t)\|$ = speed; $\|r'(t)\| = \|\vec v(t)\|$

$\vec r''(t)$ = acceleration; $\vec r''(t) = v'(t) = \vec a(t)$

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Panel 7

Ex: Suppose the path of a particle at time t is $\vec{r}(t) = \langle t^3, t^2 \rangle$. Find velocity, speed, and acceleration when $t = 1$. Illustrate.

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Panel 8

Ex: A particle starts at $P(1,0,0)$ with initial velocity $\langle 1,-1,1 \rangle$. The acceleration is $\vec{a}(t) = \langle 4t, 6t, 1 \rangle$. Find velocity, speed, and position.

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Panel 9

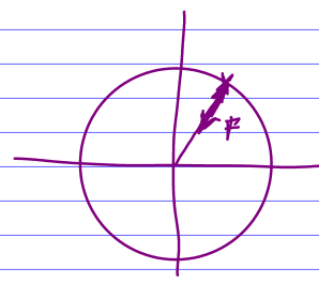
Ex: An object with mass m moves in a circle with constant angular speed ω. Find the force acting on the object and illustrate.

$r(t) = \langle \cos(\omega t), \sin(\omega t) \rangle$ circle

$r'(t) = \langle -\omega \sin(\omega t), \omega \cos(\omega t) \rangle$, $s = \omega$ ✓

$r''(t) = \langle -\omega^2 \cos(\omega t), -\omega^2 \sin(\omega t) \rangle$

$\vec{F} = m\vec{a} = -m\omega^2 r(t)$!



$\vec{F}$ points towards origin, always!

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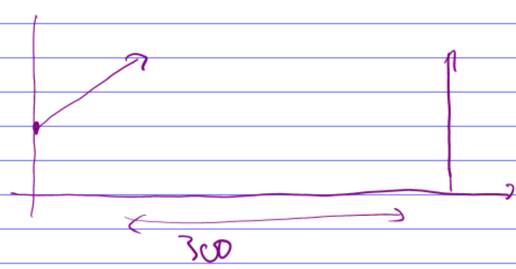
Panel 10

Application of Motion

A baseball is hit 3 feet above ground at 100 feet per second and at an angle of Pi/4 with respect to the ground. Find the maximum height reached by the baseball. Will it clear a 10-foot high fence located 300 feet from home base?



$r(0) = (0, 3)$

$v(0) = \frac{100}{\sqrt{2}}(1, 1)$

$a(t) = \langle 0, -g \rangle$

$v(t) = \langle c_1, -gt + c_1 \rangle = \langle \frac{100}{\sqrt{2}}, -gt + \frac{100}{\sqrt{2}} \rangle$

$a(t) = \langle \frac{100}{\sqrt{2}} t + d_1, -\frac{1}{2} g t^2 + \frac{100}{\sqrt{2}} t + d_2 \rangle =$

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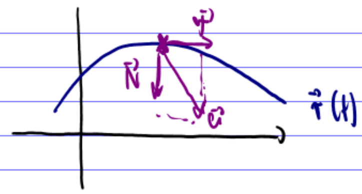
Panel 11

Tangential and Normal Components of Acceleration



acceleration can be divided
- portion in direction of $\vec{T}$: causes change in speed
- portion in direction of $\vec{N}$: causes change in direction

$\vec{a} = a_t \cdot \vec{T} + a_N \vec{N}$

a_T is tangential comp. of $\vec{a}$
a_n is normal

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Panel 12

Theorem: $\vec{a} = a_T \vec{T} + a_N \vec{N}$ where

tang. component $a_t = \frac{v \cdot a}{s}$

normal component $a_N = \frac{\|v \times a\|}{s}$

Ex: $r(t) = \langle t^2, t^2, t^3 \rangle$ - find a_T and a_N

$r'(t) = \langle 2t, 2t, 3t^2 \rangle$, $r''(t) = \langle 2, 2, 6t \rangle$, $s = \sqrt{8t^2 + 9t^4}$

$a_T = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$, $\begin{vmatrix} i & j & k \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = \ldots$

$a_N =$

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Panel 13

Quiz 4

Suppose $\vec{r}(t) = \langle t^2, 2, t \rangle$ is a vector-valued function (aka space curve), representing the position of a particle. Find the following:

1. The velocity at $P(0,0,0)$
2. The speed at $P(0,0,0)$
3. The acceleration at $P(0,0,0)$
4. The unit tangent $\vec{T}(t)$ at $P(0,0,0)$
5. The unit normal vector $\vec{N}(t)$ at $P(0,0,0)$
6. The bi-normal vector $\vec{B}(t)$ at $P(0,0,0)$
7. The curvature k at $P(0,0,0)$
8. The tangential component of the acceleration a_T at $P(0,0,0)$
9. The normal component of the acceleration a_N at $P(0,0,0)$
10. The osculating plane at $P(0,0,0)$
11. The osculating circle at $P(0,0,0)$

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