Panel 1

Last Time we discussed vector-valued functions

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. We talked about:

Tangent vectors:

Unit tangent vectors:

Length:

Curvature:

1

Panel 2

$r(t) = \langle 3\sin(t), 4t, 3\cos(t) \rangle$, start $(0, 0, 3)$

$$s = \int_0^c \|r'(t)\| \, dt = 5$$

$$\int_0^c \sqrt{9\cos^2(t) + 16 + 9\sin^2(t)} \, dt = \int_0^c \sqrt{9(\cos^2(t) + \sin^2(t)) + 16} \, dt = \int_0^c 5 \, dt = 5$$

$5c = 5$, $c = 1$

$\langle 3\sin(1), 4, 3\cos(1) \rangle$

2

Panel 3

$\langle \sqrt{2}t, e^t, e^{-t} \rangle \qquad r'(t)/\|r'(t)\|$

$\langle \sqrt{2}, e^t, -e^{-t} \rangle \cdot \dfrac{1}{\sqrt{2+e^{2t}+e^{-2t}}} = \boxed{\langle \sqrt{2}, e^t, -e^{-t} \rangle \dfrac{1}{e^t+e^{-t}}}$

$e^{2t}+2+e^{-2t} = (e^t)^2+2+(e^{-t})^2 = (e^t+e^{-t})^2$

3

Panel 4

$\langle a\cos(t), a\sin(t), bt \rangle$

$r(t) = \langle 2t, 4\sin(t), 4\cos(t) \rangle$

$\Rightarrow \quad \kappa = \dfrac{4}{20}$

$r'(t) = \langle -a\sin(t), a\cos(t), b \rangle$

$\|r'(t)\| = \sqrt{a^2\cos^2(t) + a^2\sin^2(t) + b^2} = \sqrt{a^2+b^2}$

$T = \dfrac{1}{\sqrt{a^2+b^2}} \langle -a\sin(t), a\cos(t), b \rangle$

$T' = \dfrac{1}{\sqrt{a^2+b^2}} \langle -a\cos(t), -a\sin(t), 0 \rangle$

$\|T'\| = \dfrac{1}{\sqrt{a^2+b^2}} \cdot a$

$\kappa = \dfrac{\|T'\|}{\|r'\|} = \dfrac{a}{\sqrt{a^2+b^2}} \cdot \dfrac{1}{\sqrt{a^2+b^2}} = \underline{\underline{\dfrac{a}{a^2+b^2}}}$

4

Panel 5

Name:

Quiz #4

1. Describe the curves given by

a) $\vec{r}(t) = \langle \cos(t), t, \sin(t) \rangle$

b) $\vec{r}(t) = \langle 5t, 1-t, 2t+1 \rangle$

2. If $\vec{r}(t) = \langle t \sin(t), e^{t^2} \rangle$, find $\vec{r}\,'(t)$

5

Panel 6

3. Find the length of $\langle 2t, t^2, \frac{1}{3}t^3 \rangle$ for $0 \le t \le 1$

4. Find the unit tangent to $\vec{r}(t) = \langle 3\cos(t), 5t, 3\sin(t) \rangle$

6

Panel 7

Ex. Find the curvature for $\vec{r}(t) = \langle t, t^2 \rangle$

$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|} \quad \text{so we need } r' \text{ and } T \text{ first:}$$

$$r'(t) = \langle 1, 2t \rangle \quad , \|r'(t)\| = \sqrt{1+4t^2}$$

$$T = \langle 1, 2t \rangle \cdot \left(1+4t^2\right)^{-1/2} = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle$$

$$T'(t) = -\frac{1}{2}\left(1+4t^2\right)^{-3/2} \cdot (8t) \langle 1, 2t \rangle + \left(1+4t^2\right)^{-1/2} \langle 0, 2 \rangle$$

$$= -\frac{4t}{(1+4t^2)^{3/2}} \langle 1, 2t \rangle + \frac{1}{(1+4t^2)^{1/2}} \langle 0, 2 \rangle$$

7

Panel 8

$$= -\frac{4t}{(1+4t^2)^{3/2}} \langle 1, 2t \rangle + \frac{1}{(1+4t^2)^{1/2}} \langle 0, 2 \rangle =$$

$$\frac{1}{(1+4t^2)^{1/2}} \left(\frac{-4t}{1+4t^2} \langle 1, 2t \rangle + \langle 0, 2 \rangle \right) =$$

$$\frac{1}{(1+4t^2)^{1/2}} \left[\left\langle \frac{-4t}{1+4t^2} + 0, \frac{-8t^2}{1+4t^2} + 2 \right\rangle \right] = \frac{1}{(1+4t^2)^{1/2}} \left(-\frac{4t}{1+4t^2}, \frac{\cancel{-8t^2} + 2 + \cancel{8t^2}}{1+4t^2} \right)$$

$$= \frac{1}{(1+4t^2)^{3/2}} \langle -4t, 2 \rangle$$

$$\|T'\| = \frac{1}{(1+4t^2)^{3/2}} \sqrt{16t^2+4} = \frac{2}{(1+4t^2)^{3/2}} \sqrt{4t^2+1} = \frac{2}{(1+4t^2)}$$

8

Panel 9

So far we have:

$$r(t) = \langle t, t^2 \rangle \quad , r'(t) = \langle 1, 2t \rangle \quad , \|r'(t)\| = \sqrt{1+4t^2}$$

$$T(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle$$

$$T'(t) = \frac{1}{(1+4t^2)^{3/2}} \langle -4t, 2 \rangle$$

$$\|T'(t)\| = 2/_{1+4t^2}$$

Therefore the curvature works out to be:

$$\kappa = \frac{\|T'\|}{\|r'\|} = \frac{2}{(1+4t^2)} \cdot \frac{1}{(1+4t^2)^{1/2}} = \frac{2}{(1+4t^2)^{3/2}}$$

9

Panel 10

Finding the curvature requires finding the derivative T' of the unit tangent T. Since T usually involves square roots, it is almost always painful to find T' and therefore the curvature. There is, however, a short cut:

Theorem: If $r: \mathbb{R} \to \mathbb{R}^3$, then curvature of $\vec{r}$ is

$$\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{\|T'\|}{\|r'\|}$$

10

Panel 11

In our previous example $r(t) = \langle t, t^2 \rangle$ is only a function in $\mathbb{R}^2$ so we can't apply the theorem. But,

$$\vec{r}(t) = \langle t, t^2, 0 \rangle$$

$$r'(t) = \langle 1, 2t, 0 \rangle$$

$$r''(t) = \langle 0, 2, 0 \rangle$$

$$\begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{2}{\left(\sqrt{1+4t^2}\right)^3} = \frac{2}{(1+4t^2)^{3/2}}$$

11

Panel 12

To continue we first need a theoretical result:

<u>Thm</u>: If $\vec{r}(t)$ is a space curve s.t. $\|r(t)\| = 1 \ \forall t$, then $r(t)$ and $r'(t)$ are perpendicular

<u>Proof</u>: Know $\|r(t)\| = 1 \quad |^2$

$$r(t) \cdot r(t) = \|r(t)\|^2 = 1 \quad \Big| \ \tfrac{d}{dt}$$

$$r'(t)r(t) + r(t)r'(t) = 0$$

$$2\, r(t) r'(t) = 0 \Rightarrow r \cdot r' = 0$$ i.e. they are perpendicular.

12

Panel 13

Since for the unit tangent $\vec{T}(t)$ we have $\|\vec{T}\| = 1$ we know that $\vec{T}$ and $\vec{T}'$ are perpendicular to each other.

<u>Ex</u>: For $\vec{r}(t) = \langle t, t^2 \rangle$ we had:

$$T(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle \quad , \quad T'(t) = \frac{1}{(1+4t^2)^{3/2}} \langle -4t, 2 \rangle$$

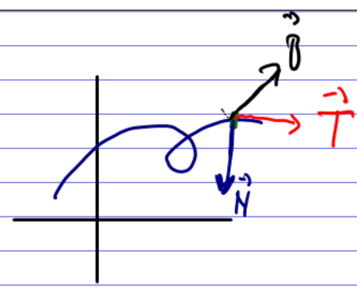
Thus $T \cdot T' = \# \langle 1, 2t \rangle \cdot \langle -4t, 2 \rangle = \underline{\underline{0}}$

We normalize the vector T' and give it a new name:

13

Panel 14

<u>Def</u>: If $\vec{T}(t)$ is the unit tangent to a space curve $\vec{r}(t)$ then $\vec{T}'(t)$ is perpendicular to $\vec{T}(t)$. We define:



$\vec{N}(t) = T'(t) / \|T'\|$ is the unit normal vector

$\vec{B}(t) = \vec{T} \times \vec{N}$ is the binormal vector

<u>Note</u>: At any point of $\vec{r}(t)$, the 3 vectors $\vec{T}, \vec{N}, \vec{B}$ are perp. to each other. They form a "<u>local coordinate system</u>"

14

Panel 15

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find tangent, unit normal and binormal vectors.

$$T = \langle -\sin(t), \cos(t), 1 \rangle \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$$

$$T' = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle \qquad (\text{note that } T \cdot T' = 0)$$

$$N = T' / \|T'\| = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle \cdot \sqrt{2} \qquad \|T'\| = 1/\sqrt{2}$$

$$= \langle -\cos(t), -\sin(t), 0 \rangle$$

$$B = T \times N = \begin{vmatrix} i & j & k \\ -\frac{1}{\sqrt{2}}\sin(t) & \frac{1}{\sqrt{2}}\cos(t) & \frac{1}{\sqrt{2}} \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} = \left\langle \frac{1}{\sqrt{2}}\sin(t), -\frac{1}{\sqrt{2}}\cos(t), \frac{1}{\sqrt{2}} \right\rangle$$

15

Panel 16

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find tangent, unit normal and binormal vectors at $t=0$

$$T(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$$

$$N(t) = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$B(t) = \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle$$

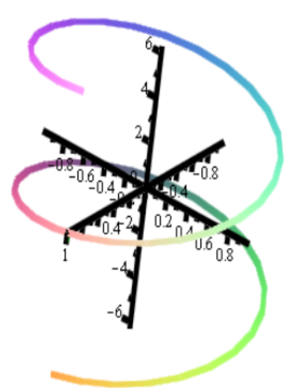
$t=0$:

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

HW



16