Panel 1

Last Time:

Vector-valued functions: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\frac{d}{dt}\vec{r}(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Integrals $\int \vec{r}(t)\,dt = \langle \int f\,dt, \int g\,dt, \int h\,dt \rangle$

Graphs of $\vec{r}(t)$ — line, slinky, spiral

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Panel 2

Let's do a Spiral

$$\langle t\cdot\cos(t), t\cdot\sin(t) \rangle$$

or a 3D Spiral: $\langle t\cos(t), t\sin(t), t \rangle$

Graph $\langle t, \sin(t) \rangle$ means:

$$x = t, \quad y = \sin(t) = \sin(x) \qquad y = \sin(x)$$

$\langle \cos(t), t \rangle$: $x = \cos(y)$

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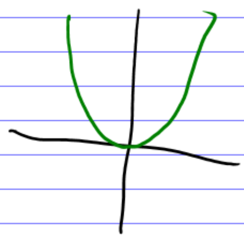
Panel 3

(1) $\langle t^3, t^6 \rangle \quad : x = t^3, \; \underline{\underline{y}} = t^6 = (t^3)^2 = \underline{\underline{x^2}}$

(2) $\langle t^2, t^4 \rangle \quad : \underline{x} = t^2, \; \underline{\underline{y}} = t^4 = (t^2)^2 = \underline{\underline{x^2}}$

(3) $\langle t, t^2 \rangle \quad : \underline{\underline{y = x^2}}$

(1)
(3)



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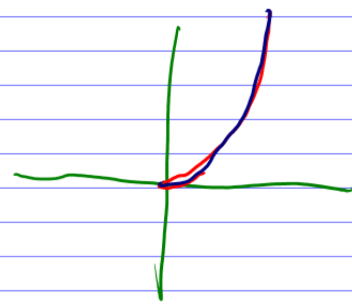

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Panel 4

$(t, t^2, t^3) \quad , \quad \langle 1+2t, 1+6t, 1+14t \rangle$

Collide?

$t = 1+2t \Rightarrow t = -1$

$t^2 = 1+6t$ ← no good. Do not collide.

$t^3 = 1+14t$

Intersect.

$t = 1+2s$

$t^2 = 1+6s \qquad \Rightarrow (1+2s)^2 = 1+6s$

$t^3 = 1+14s \qquad \cancel{1}+4s+4s^2 = \cancel{1}+6s$

$-2s+4s^2 = 0$

$-2s(1-2s) = 0, \quad s = 0, \; \frac{1}{2}$

$t = 1$, $t = 2$

yes yes

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Panel 5

Integrals of Space Curves aka vector valued functions

If $r(t) = \langle f(t), g(t), h(t) \rangle$ and f, g, h are integrable

then $\int \vec{r}(t)\,dt =$ easy

Ex: If $\vec{r}(t) = 2\cos(t)\,\vec{i} + \sin(t)\,\vec{j} + 2t\,\vec{k}$, find $\int_0^{\pi/2} \vec{r}(t)\,dt$

$$\int_0^{\pi/2} \vec{r}(t)\,dt = \left\langle 2\sin(t)\Big|_0^{\pi/2},\ -\cos(t)\Big|_0^{\pi/2},\ t^2\Big|_0^{\pi/2} \right\rangle$$

easy

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Panel 6

Arc Length (

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ then $L = \int_a^b \|r'(t)\|\,dt$

is the length of the space curve.

Note: $\int_a^b |r'(t)|\,dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$

Ex: Find length of $\vec{r}(t) = \langle r\cos(t), r\sin(t) \rangle$, $t = 0$ to 2π

$$\int_0^{2\pi} \sqrt{(r\sin(t))^2 + (r\cos(t))^2}\,dt = \int_0^{2\pi} r\,dt = 2\pi r$$

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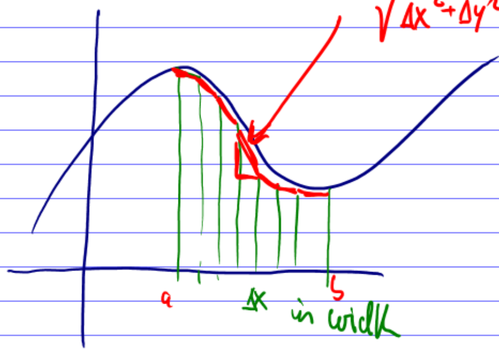
Panel 7



$$\sqrt{\Delta x^2+\Delta y^2} = \Delta t \cdot \sqrt{\frac{1}{\Delta t^2}\left(\Delta x^2+\Delta y^2\right)} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$$

$$L = \lim_{\Delta t\to 0} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t = \int_a^b \sqrt{(x')^2+(y')^2}\,dt$$

$$= \int_a^b \|\vec{r}\,'(t)\|\,dt$$

(Previously: $s = \int \sqrt{1+(f')^2}\,dx$)

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Panel 8

Ex: Find length of $r(t) = \langle \cos(t), \sin(t), t \rangle$ from $\underset{t=0}{(1,0,0)}$ to $\underset{t=2\pi}{(1,0,2\pi)}$.

$$s = \int_0^{2\pi} \|r'(t)\|\,dt = \int_0^{2\pi} \sqrt{\sin^2(t)+\cos^2(t)+1}\,dt$$

$$= \int_0^{2\pi} \sqrt{2}\,dt = \sqrt{2}\,2\pi$$

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Panel 9

It is possible for one curve to have many different parametrizations:

Ex: $r_1(t) = \langle t, t^2 \rangle$, $t = 0, 1$

Same as: $\langle t^3, t^6 \rangle$

Same as: $\langle \sin(t), \sin^2(t) \rangle$

either parametrization could be used to find, say, arc length

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Panel 10

Please note: $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ as $t = 1$ to 2 and

$t = e^u$ $\vec{r}_2(t)$ $\langle e^u, e^{2u}, e^{3u} \rangle$, $u = 0$ to $\ln(2)$

describe the same curve with different parametrization

Make sure length is the same either way:

$$s = \int_1^2 \sqrt{1^2 + (2t)^2 + (3t^2)^2}\, dt = \int_1^2 \sqrt{1 + 4t^2 + 9t^4}\, dt =$$

Let $t = e^u$: $= \int \sqrt{1 + 4e^{2u} + 9e^{4u}}\, e^u\, du$

$dt = e^u du$

$$s = \int \sqrt{(e^u)^2 + (2e^{2u})^2 + (3e^{3u})^2}\, du = \int \sqrt{e^{2u} + 4e^{4u} + 9e^{6u}}\, du = \int \sqrt{e^{2u}(1 + 4e^{2u} + 9e^{4u})}\, du =$$

$$\int \sqrt{1 + 4e^{2u} + 9e^{4u}}\, e^u\, du$$

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Panel 11

Ex: Compute length of $r(t) = \langle t, \sqrt{1-t^2} \rangle$, $t = -1$ to 1

$$S = \int_{-1}^{1} \sqrt{1^2 + \frac{t^2}{1-t^2}}\, dt = \qquad \frac{1}{2}(1-t^2)^{-1/2} \cdot (-2t)$$

$$= \int_{-1}^{1} \sqrt{\frac{1-t^2+t^2}{1-t^2}}\, dt = \int_{-1}^{1} \sqrt{\frac{1}{1-t^2}}\, dt = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, dt$$

Wait: $y = \sqrt{1-x^2} \Rightarrow y^2 + x^2 = 1$ is circle, top only.

So: $\langle \cos(t), \sin(t) \rangle$, $t = 0, \pi$

$$S = \int_0^{\pi} \sqrt{1}\, dt = \underline{\underline{\pi}}$$

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Panel 12

Def: A curve $r(t)$ is called smooth if $\vec{r}\,'(t) \neq \vec{0}$, i.e. if the components of r' are not simultaneously zero.

Def. If $\vec{r}(t)$ is a smooth curve then
$$\vec{T}(t) = \frac{1}{\|r'\|} r'(t)$$
is the unit tangent vector

Ex: $\vec{r}(t) = \langle t, t^2 \rangle$ find $T(t)$.

$\vec{r}\,'(t) = \underline{\langle 1, 2t \rangle}$ is smooth.

$$\vec{T}(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle$$

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Panel 13

Unit tangent give direction of change, normalized

$\Rightarrow$ Want to find rate of change of tangent

all these have length 1 and are tangent to the curve.

Curvature: $\chi = \left\| \frac{dT}{dt} \right\|$

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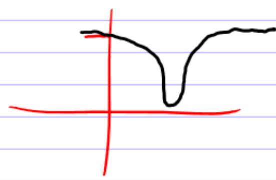

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Panel 14

We can now measure direction of curve (derivative) and length (arc length). Next we measure

Def: Curvature $\chi = \frac{\|T'(t)\|}{\|r'(t)\|}$ where T is unit tangent vector

Ex: What is the curvature of a circle with radius r.

$\vec{r}(t) = \langle R\cos(t), R\sin(t) \rangle$

$\vec{r}'(t) = \langle -R\sin(t), R\cos(t) \rangle \quad , \quad \|r'\| = R$

$T = \frac{r'(t)}{\|r'\|} = \frac{1}{R} \langle -R\sin(t), R\cos(t) \rangle = \langle -\sin(t), \cos(t) \rangle$

$T' = \langle -\cos(t), -\sin(t) \rangle \qquad \Rightarrow \chi = \frac{\|T'\|}{\|r'\|} = 1/R$

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