Panel 1

Last times

Works only in $\mathbb{R}^3$

Dot Product

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

works in any dimension

Geometrically

$\vec{a} \cdot \vec{b} = \|a\| \cdot \|b\| \cos\theta$

$a \cdot b = 0 \Leftrightarrow a \perp b$

Cross Product

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left\langle \begin{matrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{matrix} \right\rangle$$

Geometrically

$\vec{a} \times \vec{b}$

$a \times b \perp a$ and b

Projection:

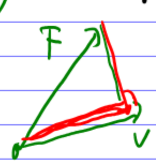
$$\text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a}$$

1

Panel 2

(3) Find the projection of $\langle 3,-1,-2 \rangle$ onto $\langle 3,3,1 \rangle$

(4) Find the work done by the force $\vec{F} = \langle 2,1,1 \rangle$ that moves an object in the direction $\langle 1,1,1 \rangle$ for 10 meters.



$$\left\| \left(\text{proj}_{\vec{v}} F \right) \right\| \cdot 10 = \left\| \frac{\vec{F} \cdot \vec{v}}{\|v\|^2} \vec{v} \right\| \cdot 10$$

$$= \frac{|F \cdot v|}{\|v\|^2} \cdot \|v\| \cdot 10 = \frac{|F \cdot v|}{\|v\|} \cdot 10 = \frac{4}{\sqrt{3}} \cdot 10$$

2

Panel 3

$\vec{a} = \langle 1,0\rangle$, $b = \langle 0,1\rangle$, $c = \langle 0,3\rangle$

$a\cdot b = 0 = a\cdot c$ but $b \neq c$

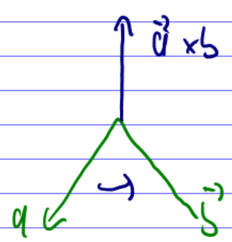
$a\cdot b = a\cdot c$ is not true, so there is no mult. inverse for dot product.

$a = \langle a_1, a_2, a_3\rangle$

$a\times b = a\times c$ is wrong also, so no mult. inverse either (Summer)

3

Panel 4



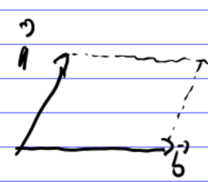
"right-hand rule" applies: as a moves onto b, thumb of right hand points in direction $\vec{a}\times\vec{b}$

parallel

$\|a\times b\| = \|a\|\|b\|\sin\theta$, i.e. if $\|a\times b\| = 0$, $a \parallel b$

$\langle 1,2,3\rangle$ and $\langle 2,4,6\rangle$ because they are scalar mult. of each other, or because $\|a\times b\| = 0$

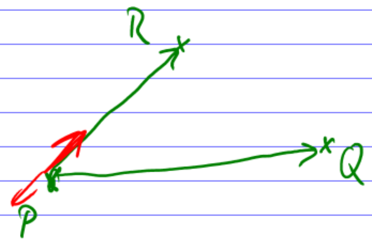
Note: $\|\vec{a}\times\vec{b}\|$ is area of parallelogram



4

Panel 5

Ex: Find vector perpendicular to the plane through $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$



$\vec{PQ} \times \vec{PR} =$

$\langle -3,1,-7\rangle \times \langle 0,-5,-5\rangle$

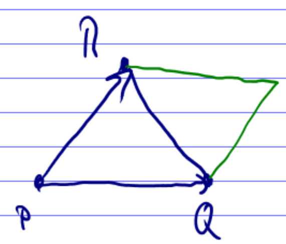
$\langle -3,1,-7\rangle \times \langle 0,1,1\rangle = \begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & 1 & 1 \end{vmatrix} = \langle 1+7, +3, -3\rangle$

$\langle 8, 3, -3\rangle$

5

Panel 6

Ex: Area of triangle $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$



$\frac{1}{2} \| \vec{PQ} \times \vec{PR} \|$

6

Panel 7

Q: Is $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$? No!

$\langle 1,0,0 \rangle \times \langle 0,1,0 \rangle = \langle 0,0,1 \rangle$

$\langle 0,1,0 \rangle \times \langle 1,0,0 \rangle = \langle 0,0,-1 \rangle$

(*) Cross product is anti-commutative

(*) Associative law does not hold either

Note: $\vec{a} \times \vec{a} \times \vec{b} = \begin{matrix} (\vec{a} \times \vec{a}) \times \vec{b} = 0 \\ \vec{a} \times (\vec{a} \times \vec{b}) \neq 0 \end{matrix}$

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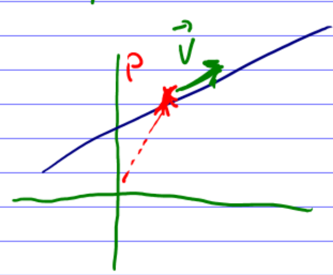
Panel 8

Of Lines and Planes $\quad ax + by + c = 0$

Line: $y = mx + b$ no good
(a) does not cover every line
(b) does not generalize to higher dim.

Think of a line as having a direction and a "starting" point P



$\ell(t) = P + t\vec{v} \quad , t \in \mathbb{R}$

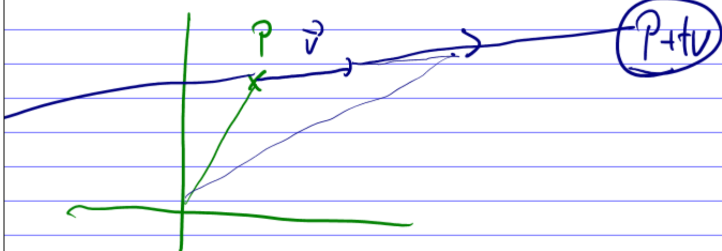
8

Panel 9

Def: If $P(x_0, y_0, z_0)$ is a point on a line, and $\vec{v} = \langle a, b, c \rangle$ is the direction of the line, then the (parametric) equation of the line is:

$$\ell(t) = P + t\vec{v}$$

$$\langle x, y, z \rangle = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle$$



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Panel 10

$y = mx + b$

$\ell = P + tv$

Ex: Find equation of a line

a) through $(5,1,3)$ and parallel to $\vec{v} = \langle 1,4,-2 \rangle$

$$\ell(t) = \langle 5,1,3 \rangle + t\langle 1,4,-2 \rangle$$

b) through $(1,2,3)$ and $(4,1,1)$

$$\ell(t) = \langle 1,2,3 \rangle + t\langle 4-1, 1-2, 1-3 \rangle =$$

$$= \langle 1,2,3 \rangle + t\langle 3,-1,-2 \rangle$$

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Panel 11

Ex: At what point does $\langle 2,4,-3\rangle + t\langle 1,-5,4\rangle$ intersect the xy-plane $\langle 2+t, 4-5t, -3+4t\rangle$

To intersect xy plane, $z=0$

$z = -3+4t = 0 \Rightarrow t = \frac{3}{4}$

$y = 4-5\left(\frac{3}{4}\right)$

$x = 2+3/4$

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Panel 12

Ex: Suppose 2 lines are $\ell_1(t) = \langle 1+t, -2+3t, 4-t\rangle$

$\ell_2(s) = \langle 2s, 3+s, -3+4s\rangle$

a) Are the lines parallel? $\ell(t) = P + t\vec{v}$

$\ell_1(t) = \langle 1,-2,4\rangle + t\langle 1,3,-1\rangle$

$\ell_2(s) = \langle 0,3,-3\rangle + s\langle 2,1,4\rangle$

b) Do they intersect in $\mathbb{R}^3$? unlikely

solving 2 equations for 2 unknowns would have to match 3rd eq.

$\ell_1(t) = \ell_2(s)$: $1+t = 2s \Rightarrow t = 2s-1$

$-2+3t = 3+s \Rightarrow -2+3(2s-1) = 3+s$ Not

$-2+6s-3 = 3+s$

$4-t = -3+4s$ $\quad -8 = -5s,\ s = \frac{8}{5}$

$7 \neq \frac{43}{5}$ $\quad 4 - \frac{11}{5} = 3 + \frac{32}{5}$ $\quad t = \frac{16}{5} - 1 = \frac{11}{5}$

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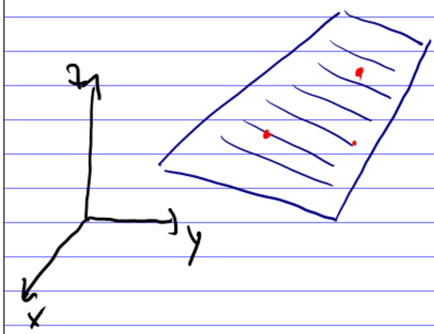
Panel 13

Planes in $\mathbb{R}^3$



A plane in $\mathbb{R}^3$ is uniquely determined by

(a) 3 points

(b) 2 vectors + 1 point

(c) 1 vector perpendicular to plane, + 1 point.

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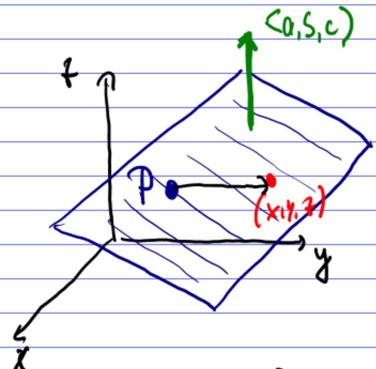
Panel 14

Suppose a plane goes through $P(x_0, y_0, z_0)$ with normal vector $\langle a, b, c\rangle \vec{n}$

perp. to plane



Take any $Q(x, y, z)$ in the plane

Then: $PQ = \langle x - x_0, y - y_0, z - z_0\rangle$ is in the plane

$$\Rightarrow \vec{n} \cdot \vec{PQ} = 0 \Rightarrow \langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle =$$

$$= a(x - x_0) + b(y - y_0) + c(z - z_0) =$$

$$= ax + by + cz + \underset{-ax_0 - by_0 - cz_0}{d} = 0$$

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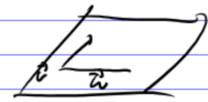
Panel 15

Def: The equation of a plane with normal vector $\vec{n} = \langle a,b,c\rangle$ through the point $P_0(x_0,y_0,z_0)$ is:

$$ax+by+cz+d=0$$



$$\iff a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Line in 2D: $ax+by+c=0$ $\qquad \ell(t)=P_0+t\vec{v}$

Plane in 3D: $ax+by+cz+d=0$ $\qquad p(s,t)=P_0+t\vec{v}+s\vec{w}$

3D object in R^4: $ax+by+cz+dw+e=0$

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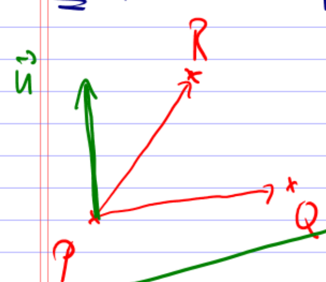
Panel 16

Scalar equation of plane through $P(x_0,y_0,z_0)$ with normal vector $\vec{n}=\langle a,b,c\rangle$ is $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

Ex: Plane through $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$



Need $\vec{n}$ = normal vector!

$$\vec{n} = \vec{PQ}\times\vec{PR} = \begin{array}{c} \\ PQ: \\ PR: \end{array}\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 8+4, -(-4-16), -2+16\rangle = \langle 12, 20, 14\rangle$$

$$\boxed{12(x-1)+20(y-3)+14(z-2)=0}$$

$$12x+20y+14z\underbrace{-100}_{d}=0$$

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