Panel 1

Contour Integration

Contour Integrals

function f

does not quite fit.

$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x')^2+(y')^2}\,dt$

$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$

$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$

vector field $\vec F$

Goal

$\int_C \vec F \cdot d\vec r$

def: $\int_C M\,dx + N\,dy$

conservative: $f(B) - f(A)$

2D, closed curve C: $\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$

1

Panel 2

Find the following integrals. Explain your reasoning.

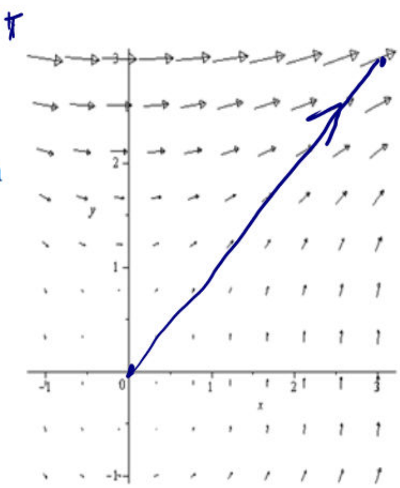
a) $\int_C \vec F \cdot d\vec r$ where $\vec F(x,y) = <\overset{M}{2xy^3 + y\sin(x)}, \overset{N}{3x^2y^2 - \cos(x)}>$ and C is the boundary of the square with corner point (0,0), (1,0), (1,1), and (0, 1), oriented counter-clockwise. 0

b) $\int_C \vec F \cdot d\vec r$ where $\vec F(x,y) = <2xy^2, 2x^2y>$ and C a curve from (0,0) to (1,1). $f = x^2y^2$ 1

c) $\int_C \overset{M}{(\cos(x^2) - 4y)}dx + \overset{N}{(e^y - 3x)}dy$, where C is the unit circle. π

d) $\int_C \vec F \cdot d\vec r$ (sign only, i.e. positive, negative, or zero) where $\vec F$ is the vector field shown and C is the line segment from (0,0) to (3,3)

positive



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Panel 3

$\overset{M}{}$ $\overset{N}{}$

Ex $\int \cos(x^2) - 4y \, dx + (e^y - 3x) dy$, C is unit circle

① Hard work : $r(t) = \langle \underbrace{\cos(t)}_{x}, \underbrace{\sin(t)}_{y} \rangle$, $t \in [0, 2\pi]$

$$\int_0^{2\pi} \left(\cos(\cos^2(x)) - 4\sin(y)\right)(-\sin(t))\,dt + \left(e^{\sin(t)} - 3\cos(t)\right)\cos(t)\,dt$$

② Green's thm applies (closed curve, 2D-vector field):

$$= \iint_{disk} \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dA = \iint_{disk} -3 - (-4) \, dA = \iint_{disk} dA = \text{area}(disk) = \pi 1^2 = \underline{\underline{\pi}}$$

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Panel 4

Alternate version of Green's Theorem:

$$\oint_C \vec{F} \, d\vec{r} = \oint_C M dx + N dy = \iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dA \qquad \underline{2D} \text{ curve}$$

Consider $F = \langle M, N \rangle = \langle M, N, 0 \rangle$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix}$$

$\Rightarrow \text{curl}(F) = \langle 0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \rangle$

Green's theorem:

$$\oint_C \vec{F} \, d\vec{r} = \iint_D \text{curl}(\vec{F}) \cdot \vec{k} \, dA \qquad (\vec{k} = \langle 0,0,1 \rangle)$$

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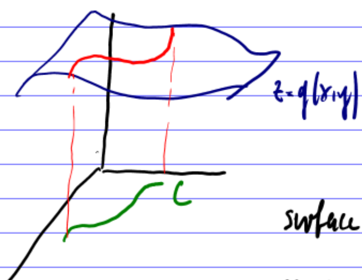
Panel 5

$y = f(x)$ a function, or $(t, f(t))$

$$\int ds = \int \sqrt{(x')^2+(y')^2}\,dt = \int_a^b \sqrt{1+(f')^2}\,dx$$

= length of curve.

$$\int_C g(x,y)\,ds = \int g(x, f(x))\sqrt{1+(f')^2}\,dx$$



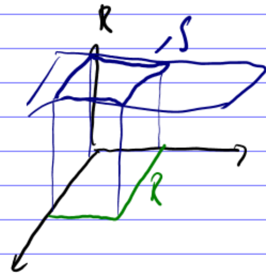
integral of that portion of surface over curve C

$z = f(x,y)$ a surface

$$\iint dS = \iint_R \sqrt{1+f_x^2+f_y^2}\,dA$$

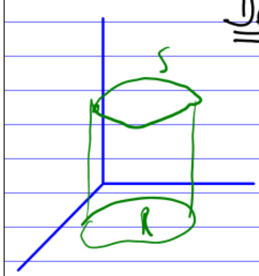
= surface area of S

$$\iint g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\,dA$$



integral over that portion of 4D-object $w = g(x,y,z)$ over surface S

Panel 6



Def: Suppose surface S is defined by $z = f(x,y)$, $(x,y) \in R$. Then

$$\iint_S \underset{w}{g(x,y,z)}\,dS = \iint_R g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\,dA$$

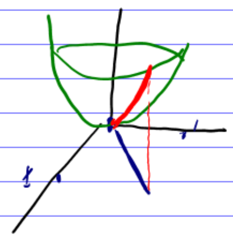
$\iint_S g(x,y,z)\,dS$ is the integral of that portion of $w = g(x,y,z)$ over surface S

where S is a surface defined on R

Panel 7

Ex: $\int_\gamma (x^2+y^2)\,ds$, $y=x$, $x\in[0,1]$

$z=x^2+y^2$



$$\int_\gamma x^2+y^2\,ds = \int_0^1 (x^2+x^2)\sqrt{1+1}\,dx = \int_0^1 2x^2\sqrt{2}\,dx = 2\sqrt{2}\,\frac{1}{3}$$

$$\int_C g(x,y)\,ds = \int_a^b g(x,f(x))\sqrt{1+(f')^2}\,ds$$

$y=f(x)$

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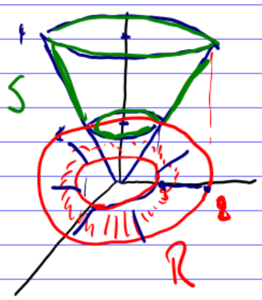
Panel 8

Ex: Evaluate $\iint_S x^2 z\,dS$, S portion of $z^2=x^2+y^2$ between $z=1$ and $z=4$.

$z=(x^2+y^2)^{1/2}$ $f_x=\frac{1}{2}(\ )^{-1/2}2x$

$$(*)\ \iint_S x^2 z\,dS = \iint_{\boxed{R}} x^2(x^2+y^2)^{1/2}\sqrt{1+f_x^2+f_y^2}\,dA$$



$$1+f_x^2+f_y^2=\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1=2$$

$$(*)=\iint_R x^2\sqrt{x^2+y^2}\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_1^2 r^2\cos^2 t\cdot r\ r\,dr\,d\theta$$

cone

$z^2=x^2+y^2$ $1=x^2+y^2$, or $4=x^2+y^2$, or $9=x^2+y^2$

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Panel 9

Surface integrals have functions as integrand. Want to extend that integral to vector fields:

Def: If S is surface given by $z = f(x,y)$, over region R.

$\Rightarrow \iint_S \vec{F} \cdot \vec{n} \, dS$ is the flux of the vector field

where $\vec{n}$ is the normal vector to the surface S

given by $\vec{n} = \langle -f_x, -f_y, 1 \rangle \cdot \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}}$

grad

If S is surface submerged in a vector field $\vec{F}$, then flux integral gives amount of fluid flowing through S

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Panel 10

Ex: Let S be $z = 9 - x^2 - y^2$, $z \geq 0$ and $\vec{F} = \langle 3x, 3y, z \rangle$.

Find flux of $\vec{F}$ through S. try this at home!

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