Panel 1

Conservative Vector Fields

Exam 3: next Monday 26th

$\vec{F}$ conservative $\Rightarrow \nabla f = F \Leftrightarrow curl(F) = 0$ or $M_y = N_x$

Fundamental Theorem of Line Integration: $\vec{F}$ conservative

$\Rightarrow \int_C \vec{F}\, d\vec{r} = f(B) - f(A)$, A start of C, B is end of C

$\Rightarrow \int_C \vec{F}\, d\vec{r}$ is path independent

$\Rightarrow \oint_C \vec{F}\, d\vec{r} = 0$

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Panel 2

Find a conservative vector field that has the given potential:

$f(z, y, z) = \sin(x^2 + y^2 + z^2)$

Find $div(\nabla \cdot F)$ and $curl(F) = \nabla \times F$

$F(x, y, z) = < x^2 z, y^2 x, y + 2z >$

Evaluate $\int_C (x - y)dx + xdy$ if C is the graph of $y^2 = x$ from (4,-2) to (4,2)

Find the work done by F(x,y,z) along the curve $< t, t^2, t^3 >$ from (0, 0, 0) to (2, 4, 8), where $F(x, y, z) = < y, z, x >$

Check which of the following vector fields is not conservative.

$F(x, y) = < 3x^2 y + 2, x^3 + 4y^3 >$

$F(x, y) = < e^x, 3 - e^x \sin(y) >$

$F(x, y, z) = < 8xz, 1 - 6yz^2, 4x^2 - 9y^2 z^z >$

Show that the line integrals are independent of the path, and find their value:

$$\int_{(-1,2)}^{(3,11)} (y^2 + 2xy)dx + (x^2 + 2xy)dx$$

$$\int_{(1,0,2)}^{(-2,1,3)} (6xy^3 + 2z^2)dx + (9x^2 y^2)dy + (4xz + 1)dz$$

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Panel 3

$\int_C \vec{F}\, d\vec{r}$ — long : $\int M(x(t), y(t))\, dx + N\, dy + P\, dz$

— short : $f(B) - f(A)$ conservative

$\oint_C \vec{F}\, d\vec{r}$ — long

— zero if $\vec{F}$ is conservative

— $\boxed{\text{Green's thm}}$

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Panel 4

<u>Green's Theorem.</u> R a region in xy-plane with boundary curve C. C is piecewise smooth, non-intersecting, closed, and positively oriented. $\vec{F} = (M, N)$ is a smooth vector field. Then:

$$\oint_C \vec{F}\, d\vec{r} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Note : if $\vec{F}$ is conservative:

$$\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA = 0$$

$\boxed{\text{2D Thm!}}$ There is a 3D version – Bonus

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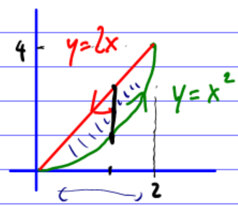
Panel 5

Ex: Evaluate $\oint_C 5xy\,dx + x^3\,dy$, where C is as shown:



Method A: old way:

$$\int_{y=x^2} 5xy\,dx + x^3\,dy + \int_{y=2x} 5xy\,dx + x^3\,dy =$$

$$= \int_0^2 5t\cdot t^2\,dt + t^3\,2t\,dt + \int_2^0 5t\,2t\,dt + t^3\,2\,dt = -\tfrac{28}{15}$$

Method B: Green's theorem

$$\int 5xy\,dx + x^3\,dy = \iint_R 3x^2 - 5x\,dA = \int_0^2\int_{x^2}^{2x} 3x^2 - 5x\,dy\,dx =$$

$$\int_0^2 3x^2y - 5xy\Big|_{y=x^2}^{y=2x} dx = \ldots = -\tfrac{28}{15}$$

Panel 6

Ex: Evaluate $\oint_C 2xy\,dx + (x^2+y^2)\,dy$, C is $4x^2 + 9y^2 = 36$

Old way: $r(t) = \langle \frac{?}{6}\cos(t), \frac{3}{6}\sin(t)\rangle$ maybe →

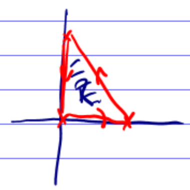
Green's theorem: $\oint_C 2xy\,dx + (x^2+y^2)\,dy = \iint_R 2x - 2x\,dA = 0$

way easier

Panel 7

Ex: Find $\oint_\gamma \overset{M}{(x\sin(y^2)-y)}\,dx + \overset{N}{(x^2 y\cos(y^2)+3x)}\,dy$ (*)

where γ is the triangle $(0,0)$, $(1,0)$, $(0,1)$.



Old way: 3 integrals

New way: (*) $= \iint_R 2x\cancel{y\cos(y^2)} + 3 - 2\cancel{xy\cos(y^2)} - 1\,dA =$

$= \iint_R 2\,dA = 2\iint_R dA = 2\,\text{area}(R) = 2\cdot\frac{1}{2} = \underline{\underline{1}}$

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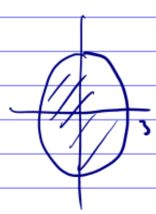
Panel 8

Evaluate $\oint_C \overset{M}{(3y - e^{\sin(x)})}\,dx + \overset{N}{(7x + \sqrt{y^4+1})}\,dy$ (*)

where C is the circle $\underline{x^2+y^2=9}$

Old way: one integral, but $x(t)=3\cos(t)$, $y(t)=3\sin(t)$

Green's thm: (*) $= \iint_R 7-3\,dA = 4\iint_R dA = 4\cdot\pi 9 = \underline{\underline{36\pi}}$



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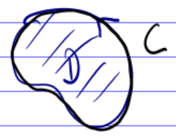
Panel 9

Theorem: If D is a region enclosed by a curve C then $\text{area}(D) = \frac{1}{2}\oint_C x\,dy - y\,dx$



Proof: $\oint_C \overset{N}{x}\,dy \overset{M}{-y}\,dx = \iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$

$= \iint_D 1 - (-1)\,dA =$

$= 2\iint_D dA = 2\,\text{area}(D)$

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Panel 10

Ex: Find area enclosed by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

area of ellipse: $r(t) = \langle \underset{X(t)}{a\cos(t)},\ \underset{Y(t)}{b\sin(t)} \rangle,\ t \in [0, 2\pi]$

$A = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} ab\cos^2(t)\,dt + ba\sin^2(t)\,dt =$

$= \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{1}{2}2\pi ab = \pi ab$

$\Rightarrow$ area of circle: πr^2 $(a = b = r)$

Note: Circumf. of ellipse is HARD

length of circle: $\text{length} = \int_0^{2\pi}\sqrt{(x')^2 + (y')^2}\,dt = \int_0^{2\pi}\sqrt{R^2}\,dt = R\cdot 2\pi = 2\pi R$

$r(t) = \langle R\cos(t), R\sin(t) \rangle$

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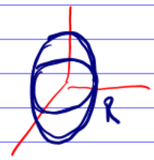
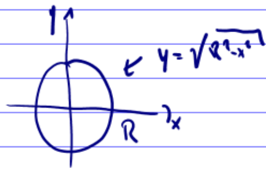
Panel 11

Review: Volume of sphere radius R?

$$x^2+y^2+z^2=R \Rightarrow z=\pm\sqrt{R^2-x^2-y^2}$$



$$V=2\iint_{\text{circle}}\sqrt{R^2-x^2-y^2}\,dA=2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}}\sqrt{R^2-x^2-y^2}\,dy\,dx$$



$$=2\int_0^{2\pi}\int_0^R\sqrt{R^2-r^2}\;r\,dr\,d\theta=$$

$x=r\cos(t)$

$y=r\sin(t)$

$$=2\cdot 2\pi\left[-\frac{1}{3}(R^2-r^2)^{3/2}\Big|_0^R\right]=2\cdot-\frac{2}{3}\pi\left(-R^3\right)=\frac{4}{3}\pi R^3$$

Panel 12

More Review. Surface area of a sphere radius R?

$$f(x,y)=z=\sqrt{R^2-x^2-y^2}\qquad S=\iint_R dS=\iint_R\sqrt{1+f_x^2+f_y^2}\,dA$$

$$f_x=\left(\tfrac{1}{2}\right)\cdot 2x\left(R^2-x^2-y^2\right)^{-1/2}=-\frac{x}{\sqrt{R^2-x^2-y^2}}$$

$$f_y=-\frac{y}{\sqrt{R^2-x^2-y^2}}\quad\Rightarrow\quad\iint_R dS=\iint\sqrt{1+f_x^2+f_y^2}=4\pi r^2$$

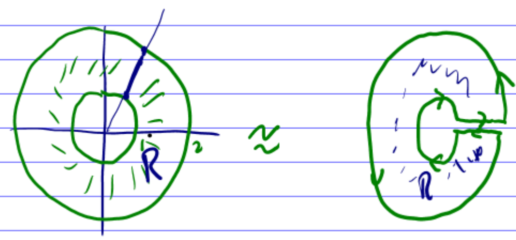
Interesting

	Vol	Area
$\mathbb{R}^2$	πr^2	$2\pi r$
$\mathbb{R}^3$	$\frac{4}{3}\pi r^3$	$4\pi r^2$
$\mathbb{R}^4$		

$\boxed{\text{Final project!}}$

Panel 13

Ex: Evaluate $\oint_C \overset{M}{y^2}\,dx + \overset{N}{3xy}\,dy$ where C is the boundary of the region between $x^2+y^2=1$ and $x^2+y^2=4$.



Green's thm applies

$x = r\cos(t)$
$y = r\sin(t)$ $\quad dA = r\,dr\,d\theta$

$$\oint_C y^2\,dx + 3xy\,dy = \iint_R 3y - 2y\,dA = \iint_R y\,dA = \int_0^{2\pi}\int_1^2 r\sin(\theta)\,r\,dr\,d\theta =$$

$=$ easy!

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Panel 14

15. Evaluate $\int_C 2(x+y)dx + 2(x+y)dy$, C curve from (-2, 2) to (4, 3)

Find potential $\rightarrow f(B) - f(A)$

16. Find the work done by the force field $F = <9x^2y^2, 6x^3y-1>$ from P(0,0) to Q(5,9)

$\int 9x^2y^2\,dx + 6x^3y - 1\,dy$

find potential, $f(B) - f(A)$

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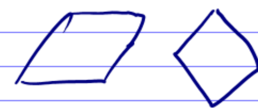
Panel 15

18. Evaluate $\oint_C 2xy\,dx + (x+y)\,dy$ where C bounds the region between $y=0$ and $y = 4 - x^2$.

long way or Green's thm

21. Evaluate $\oint_C x\sin(y^2) - y^2)dx + (x^2\cos(y^2) + 3x)dy$ where C is the boundary of the trapezoid with vertices (0, -2), (1, -1), (1, 1), and (0, 2).

long way $\Rightarrow$ ~~4~~ integrals



$\boxed{\text{Green's Thm}}$

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