Panel 1

Last Time:

Finding potential functions for conservative vector fields $\vec{F} = \langle M, N \rangle$ or $\vec{F} = \langle M, N, P \rangle$

Line Integral of a function with respect to arc length

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\,dt$$

Line Integral of a function with respect to x or y

$$\int_C f(x,y)\,dx = \int f(x(t), y(t))\, x'(t)\,dt, \quad \int_C f(x,y)\,dy = \int f(x(t), y(t))\, y'(t)\,dt$$

Integral of a vector field $\vec{F} = \langle M, N, P \rangle$

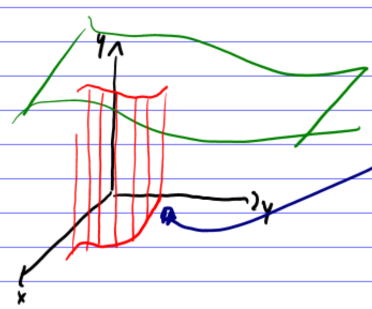
$$\int_C \vec{F}\,d\vec{r} = \int_C M\,dx + N\,dy + P\,dz$$ Work

1

Panel 2

$\int \vec{F}\,d\vec{r}$ is work

$\int_C f(x,y)\,ds$ area of "curtain" over C and $f(x,y)$



2

Panel 3

Integral Soup

area of "curtain"

$\int_a^b f(x)\,dx = F(b) - F(a)$ curve area

$\int_C f(x,y)\,ds = \int f(x(t),y(t))\sqrt{x'^2+y'^2}\,dt$

$\iint_R f(x,y)\,dA$ = Fubini volume

$\int_C f(x,y)\,dx = \int f(x(t),y(t))\,x'(t)\,dt$

$\iiint_Q f(x,y,z)\,dV$ = iterated integral

$\int_C f(x,y)\,dy = \ldots\; y'(t)\,dt$

$\int_a^b ds = \int_a^b \sqrt{x'(t)^2+y'(t)^2}\,dt$ or $\int_a^b \sqrt{1+f'(x)^2}\,dx$ length

$\int_C \vec{F}\,d\vec{r} = \int \langle M,N\rangle \langle dx,dy\rangle = \int M\,dx + N\,dy$

$\iint_R dS = \iint_R \sqrt{f_x^2+f_y^2+1}\,dA$ surface area

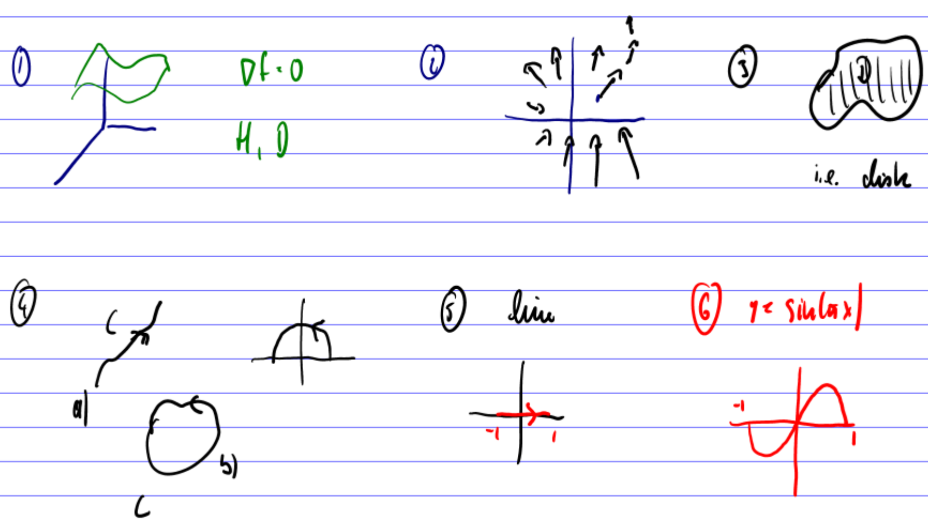
3

Panel 4

Let $f(x,y) = x^2 - xy + y^2$, $F(x,y) = \langle 2x - y, 2y - x\rangle$, $D = \{(x,y) : x^2 + y^2 \le 1\}$,

$C = \{(x,y) : x^2 + y^2 = 1, y \ge 0\}$, $\gamma_1(t) = \langle t, 0\rangle$, $t \in [-1,1]$, and $\gamma_2(t) = \langle t, \sin(\pi t)\rangle$, $t \in [-1,1]$.

Sketch each object



4

Panel 5

Let $f(x,y) = x^2 - xy + y^2$, $F(x,y) = <2x - y, 2y - x>$, $D = \{(x,y): x^2 + y^2 \le 1\}$,
$C = \{(x,y): x^2 + y^2 = 1, y \ge 0\}$, $\gamma_1(t) = <t, 0>$, $t \in [-1,1]$, and $\gamma_2(t) = <t, \sin(\pi t)>$, $t \in [-1,1]$.

a) $\cancel{\int_D f(x,y)dA}$ $\iint_D f(x,y)\,dA$

b) $\cancel{\int_D f(x,y)ds}$ $\int_C f(x,y)\,ds$

c) $\int_C f(x,y)ds$ ✓ $x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 1$

d) $\int_{\gamma_1} f(x,y)dx$ ✓

e) $\int_{\gamma_2} f(x,y)dy$ ✓

f) $\cancel{\int_{\gamma_1} f(x,y)d\vec{r}}$

g) $\cancel{\int_{\gamma_1} F(x,y)dx}$

h) $\cancel{\int_D F(x,y)dr}$

i) $\int_C F(x,y)dr$ ✓

j) $\int_{\gamma_1} F(x,y)dr$ ✓

k) $\int_{\gamma_2} F(x,y)dr$ ✓

5

Panel 6

Let $g(x) = x^2 + e^x$, $f(x,y) = x y e^{x^2}$, $\vec{F}(x,y) = \langle xy, x^2 - y^2 \rangle$

$C: r(t) = \langle 2t+1, t^3 - 3 \rangle$, $t \in [0,1]$, $R = [0,1] \times [0,1]$. Setup:

1. Area under g from 0 to 1

$$\int_0^1 g(x)\,dx = \int_0^1 x^2 + e^x\,dx$$

4. Surface area of f over R

$$\int_0^1\int_0^1 dS = \iint \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

2. Length of curve C

$$\int ds = \int_0^1 \sqrt{4 + 9t^4}\,dt$$

5. Work through $\vec{F}$ along C

$$\int \vec{F}\,d\vec{r} = \int_C xy\,dx + (x^2 - y^2)\,dy$$

3. Volume under f over R

$$\int_0^1\int_0^1 x y e^{x^2}\,dx\,dy$$

6. Area of "curtain" over C under f

$$\int_C f(x,y)\,ds$$

6

Panel 7

Ex: Evaluate $\int_C xy\,dx + x^2\,dy$ if

C_1: $\langle 3t-1, 3t^2-2t \rangle$, $1 \le t \le 5/3$ (from (2,1) to (5,4))

C_2: line segment from (2,1) to (4,5)

(Maple) tool

$$\int \vec{F}\,d\vec{r} = \int \langle xy, x^2 \rangle \langle dx, dy \rangle =$$

$$(1) \quad = \int_{C_1} xy\,dx + \int_{C_1} x^2\,dy = \int_1^{5/3} (3t-1)(3t^2-2t)\,3\,dt + \int_1^{5/3} (3t-1)^2 (6t-2)\,dt = \#$$

$$(2) \quad = \int_{C_2} xy\,dx + \int_{C_2} x^2\,dy = \int_0^1 (2+2t)(1+4t)\,2\,dt + \int_0^1 (2+2t)^2\,4\,dt = \#$$

$$C_2: \ l(t) = (2,1) + t\big((4,5)-(2,1)\big) = \langle 2,1\rangle + \langle 2,4\rangle t = \langle \underset{x}{2+2t}, \underset{y}{1+4t} \rangle, \ t \in [0,1]$$

$$A + t(B-A)$$

7

Panel 8

Fundamental Theorem of Line Integration

Suppose $\vec{F} = (M, N)$ is a conservative vector field. Then

$$\int_C \vec{F}\,d\vec{r} = f(b) - f(a), \quad f \text{ is potential function}$$

$$(\nabla f = F \text{ i.e. } f_x = M, f_y = N)$$

$$\int \vec{F}\,d\vec{r} = \int M\,dx + N\,dy = \int f_x\,dx + f_y\,dy$$

$$= \int \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}\,dt + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\,dt =$$

$$= \int \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right)dt = \int_a^b \frac{d}{dt} f(x(t), y(t))\,dt =$$

$$= f(b) - f(a)$$

8

Panel 9

<u>Ex:</u> Find work done by gravitational field $\vec{F}(\vec{r}) = -\dfrac{mMG}{\|r\|^3}\vec{r}$ moving particle from $(3,4,12)$ to $(2,2,0)$.

$\circ m$

M

$$F(x,y,z) = -\frac{mMG}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$$

$$\int_{(3,4,12)}^{(2,2,0)} \vec{F}\, d\vec{r} = f(2,2,0) - f(3,4,12) = -mMG\left(\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{169}}\right)$$

where $f(x,y,z) = -mMG\,(x^2+y^2+z^2)^{1/2}$

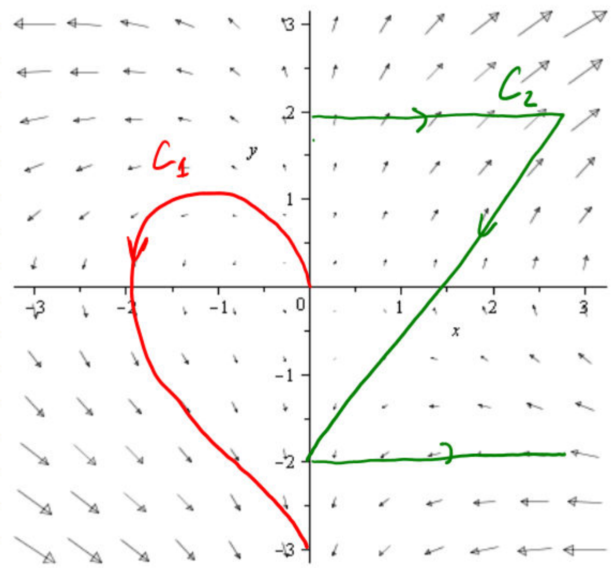
9

Panel 10

10

Panel 11

Ex: Are the following integrals positive or negative?



$\int_{C_1} \vec{F}\, d\vec{r}$ pos or neg

$\int_{C_2} \vec{F}\, d\vec{r}$ pos or neg

11

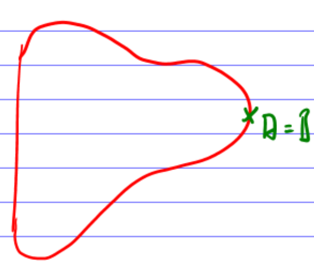
Panel 12

Corollary 2: If $\vec{F}$ is conservative and C a closed curve then

$$\int_C \vec{F}\, d\vec{r} = 0$$

Note: if C is closed we write $\oint_C \vec{F}\, d\vec{r}$

Proof $\int \vec{F}\, d\vec{r} = f(B) - f(A) = 0$



12

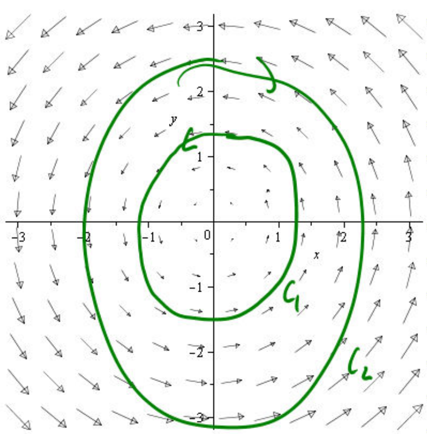
Panel 13

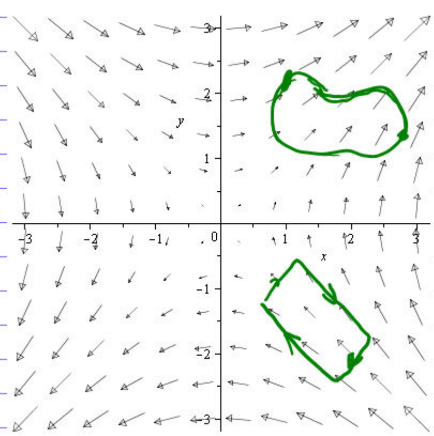
Which of the following vector fields looks conservative?

Republicans Democrats

Panel 14

Which of the following vector fields looks conservative?





not conservative

$\oint_{C_1} \vec{F}\,d\vec{r} > 0, \quad \oint_{C_2} \vec{F}\,d\vec{r} < 0$

conservative $\oint_C \vec{F}\,d\vec{r} = 0$

(hard to see)

Panel 15

HW

take home quiz on Monday

Find a conservative vector field that has the given potential:

$$f(z,y,z)=\sin(x^2+y^2+z^2)$$

Find $div(\nabla\cdot F)$ and $curl(F)=\nabla\times F$

$$F(x,y,z)=<x^2z,y^2x,y+2z>$$

Evaluate $\int_C (x-y)dx+xdy$ if C is the graph of $y^2=x$ from (4,-2) to (4,2)

Find the work done by F(x,y,z) along the curve $<t,t^2,t^3>$ from (0, 0, 0) to (2, 4, 8), where $F(x,y,z)=<y,z,x>$

Check which of the following vector fields is not conservative.

$$F(x,y)=<3x^2y+2,x^3+4y^3>$$

$$F(x,y)=<e^x,3-e^x\sin(y)>$$

$$F(x,y,z)=<8xz,1-6yz^2,4x^2-9y^2z^z>$$

Show that the line integrals are independent of the path, and find their value:

$$\int_{(-1,2)}^{(3,11)}(y^2+2xy)dx+(x^2+2xy)dx$$

$$\int_{(1,0,2)}^{(-2,1,3)}(6xy^3+2z^2)dx+(9x^2y^2)dy+(4xz+1)dz$$

15