Panel 1

Last Time:

$\vec{F} = \langle M, N \rangle$ conservative $\Rightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ / f s.t. $\nabla f = F$

$\vec{F} = \langle M, N, P \rangle$ conservative $\Rightarrow$ $\text{curl}(F) = \nabla \times F = 0$

How to find potential functions

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\, dt$$

line integral

C a curve $(x(t), y(t))$, $t \in [a,b]$

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Panel 2

Find potential for $F(x,y) = \langle y\cos(x) + 2xy,\ \sin(x) + x^2 \rangle$

(1) $\frac{\partial M}{\partial y} = \cos(x) + 2x$, $\frac{\partial N}{\partial x} = \cos(x) + 2x$ $\Rightarrow$ there is potential f:

(2) $f_x = y\cos(x) + 2xy \Rightarrow f = y\sin(x) + yx^2 + C(y)$

$f_y = \cancel{\sin(x)} + \cancel{x^2} + C'(y) = \cancel{\sin(x)} + \cancel{x^2} \Rightarrow C'(y) = 0 \Rightarrow C = \text{const}$

$f(x,y) = y\sin(x) + yx^2 + C$

Quiz question (1)

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Panel 3

Let $F(x,y,z) = \langle y+z,\ x+z,\ y+x \rangle$. Find potential:

① curl(F) toss-up

② $f_x = y+z \Rightarrow f = xy + xz + C(y,z)$

$f_y = x + C_y(y,z) = x+z \Rightarrow C_y = z \Rightarrow C = zy + C(z)$

$f = xy + xz + zy + C(z)$

$f_z = x + y + C_z(z) = y + x \Rightarrow C_z = 0 \Rightarrow C = \text{const}$

$\boxed{f(x,y,z) = xy + xz + zy + C}$

2nd quiz question

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Panel 4

Occasionally it is possible to guess a potential function:

Gravity field: $\vec{F} = \left\langle \dfrac{-x}{(x^2+y^2+z^2)^{3/2}},\ \dfrac{-y}{(x^2+y^2+z^2)^{3/2}},\ \dfrac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$

Potential function: $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

$f_x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$ ✓

$f_y =$ ✓

$f_z =$ ✓

2 quiz question ?

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Panel 5

Ex: Find $\int_C x y^2 ds$, where C describes first quarter circle, radius 1.

$$\int_C \underline{\quad} ds = \int \underset{x(t),\, y(t)}{\uparrow} \sqrt{(x')^2 + (y')^2}\, dt$$

① Find parameter equation for C : $(x(t), y(t)) = (\qquad)$

$$r(t) = \langle \underset{x}{\cos(t)}, \underset{y}{\sin(t)} \rangle,\ t \in [0, \tfrac{\pi}{2}] \checkmark$$

② $\int x y^2 ds = \int_0^{\pi/2} \cos(t) \sin^2(t) \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt =$

$$= \int_0^{\pi/2} \cos(t)\sin^2(t)\, dt = \frac{1}{3}\sin^3(t)\Big|_0^{\pi/2} = \frac{1}{3}$$

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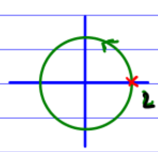
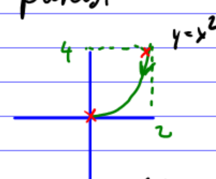
Panel 6

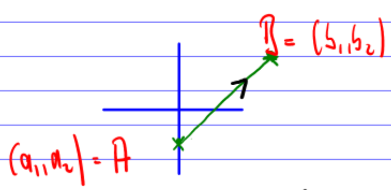
Before we continue, need to find paths: Find $r(t) = \langle x(t), y(t) \rangle$ expressions for these paths:

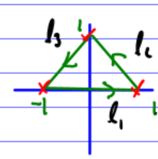


$(2\cos(t), 2\sin(t))$



$\langle t, t^2 \rangle$, $t \in [2, 0]$





$r(t) = A + t(B - A)$, $t \in [0, 1]$

$l_1(t) = (-1 + 2t, 0)$ $t \in [0, 1]$

or $l_1(t) = (t, 0)$, $t \in [-1, 1]$

l_3 similar

$(1 - t, 0 + t) = (1, 0) + t(-1, 1) = l_2(t) = (1, 0) + t[(0, 1) - (1, 0)] =$

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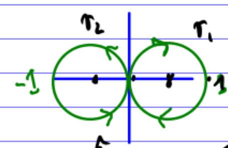
Panel 7

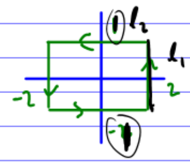
Before we continue, need to find paths: Find $r(t) = \langle x(t), y(t) \rangle$ expressions for these paths:



$$r_1 = \langle \underbrace{\tfrac{1}{2}\cos(t) + \tfrac{1}{2}}_{x},\ \underbrace{\tfrac{1}{2}\sin(t)}_{y} \rangle \qquad t \in [0, 2\pi]$$

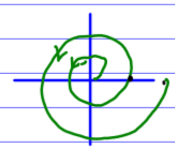
$$\left(x - \tfrac{1}{2}\right)^2 + (y)^2 = \tfrac{1}{4}$$

$$r_2(t) = \langle \tfrac{1}{2}\cos(t) - \tfrac{1}{2}, \tfrac{1}{2}\sin(t) \rangle$$



$$l_1 = \langle 2, t \rangle, \ t \in [-1, 1]$$

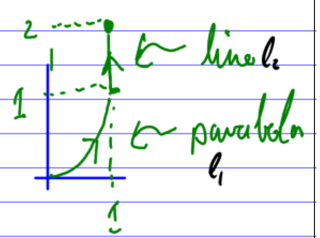
$$l_2 = \langle t, 1 \rangle, \ t \in [2, -2]$$



$$r(t) = \langle t\cos(t), t\sin(t) \rangle, \ t \in [0, 4\pi]$$

Panel 8

Ex: Evaluate $\int_C 2x \, ds$ where C:



$$l_1(t) = \langle \overset{x}{t}, \overset{y}{t^2} \rangle, \ t \in [0, 1]:$$

$$\int_C 2x \, ds = \int_0^1 2 \cdot t \cdot \sqrt{1 + 4t^2} \, dt = \frac{1}{6}\left(1 + 4t^2\right)^{3/2} \Big|_0^1$$

$$l_2(t) = \langle 1, t \rangle, \ t \in [1, 2]$$

$$\int_C 2x \, ds = \int_1^2 2 \cdot \sqrt{1} \, dt = 2t \Big|_1^2 = 2$$

+

answer

Panel 9

Question: What is $\int_C f(x,y)\,ds$ if

a) C goes along x-axis from a to b:

b) C goes along y-axis from c to d

a) $l_1(t) = \langle t, 0 \rangle$: $\int f(t,0) \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int f(t,0)\, \overbrace{x'(t)}^{1}\, dt$

b) $l_2(t) = \langle 0, t \rangle$ $\int f(0,t)\, y'\, dt = \int f(0,t) \frac{\partial y}{\partial t}\, dt$

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Panel 10

We also define two variations of line integrals:

$\int_C f(x,y)\,dx \overset{\text{def}}{=} \int_a^b f(x(t),y(t)) \overbrace{\frac{dx}{dt}}^{x'}\, dt$ line int. with respect to x

$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t)) \overbrace{\frac{dy}{dt}}^{y'}\, dt$ with respect to y

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Panel 11

Ex: Find $\int_C xy^2\,dx$ and $\int_C xy^2\,dy$ where C is parabola from (0,0) to (2,4)

C: $r(t) = \langle t, t^2 \rangle$, $t \in [0,2]$

$$\int_C xy^2\,\underline{dx} = \int_0^2 t \cdot t^4 \, x'(t)\,dt = \int_0^2 t \cdot t^4 \, 1\,dt = \int_0^2 t^5\,dt = \_\_$$

$$\int_C xy^2\,dy = \int_0^2 t\,t^4 \cdot 2t\,dt = \int_0^2 2t^6\,dt = \_\_$$

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Panel 12

Line Integrals of Vector Fields:

Suppose F is a vector field on a smooth curve C, defined via $\vec{r}(t)$, $a \le t \le b$. Then the line integral of F along C is:

Work done moving particle along C through field F

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b M\,dx + N\,dy = \int_a^b M\,dx + \int_a^b N\,dy$$

Note: $\int \vec{F}\,d\vec{r} = \int \langle M, N \rangle \cdot \langle dx, dy \rangle = \int M\,dx + N\,dy =$

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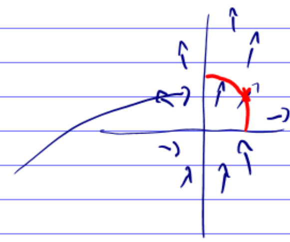
Panel 13

Ex: Let $\vec{F}(x,y) = \langle x^2, -xy \rangle$ C quarter circle radius 1.

Find $\boxed{\int_C \vec{F}\, d\vec{r}}$



$C: r(t) = \langle \cos(t), \sin(t) \rangle, t \in [0, \pi/2]$

$$\int_C \vec{F}\, d\vec{r} = \int_C M\,dx + N\,dy =$$

$x = \cos(t),\ x' = -\sin(t)$

$y = \sin(t),\ y' = \cos(t)$

$$= \int_C x^2\,dx - \int_C xy\,dy =$$

$$= \int_0^{\pi/2} \cos^2(t)(-\sin(t))\,dt - \int_0^{\pi/2} \cos(t)\sin(t)\cos(t)\,dt = -2\int_0^{\pi/2} \cos^2(t)\sin(t)\,dt$$

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Panel 14

$$-2\int_0^{\pi/2} \cos^2(t)\sin(t)\,dt = +2 \cdot \frac{1}{3}\cos^3(t)\Big|_0^{\pi/2} = -\frac{2}{3}$$

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Panel 15

Physics Interpretation of Line Integral

$$\int_C \vec{F}\, d\vec{r} = \int_a^b M\,dx + N\,dy + P\,dz$$

$$= \int_a^b M \frac{dx}{dt}\,dt + N \frac{dy}{dt}\,dt + P \frac{dz}{dt}\,dt$$

is the work done to move a particle through the field F along path C.

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