Panel 1

Last Time:

$F = (M, N, P)$

Vector fields: $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ or $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ $\vec{F}(x,y,z) = \langle y, x, xz \rangle$

$\mathrm{div}(\vec{F}): \nabla \cdot F = M_x + N_y + P_z = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle M, N, P \rangle$

Ex $\mathrm{div}(F) = x$

$$\mathrm{curl}(\vec{F}) = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \langle P_y - N_z, -(P_x - M_z), N_x - M_y \rangle$$

Conservative vector field $\vec{F}$ is a vector field for which there is a function f (potential) s.t. $\nabla f = \vec{F}$, i.e.

$f_x = M, f_y = N, f_z = P$

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Panel 2

Warm-up: Show that $\nabla \cdot (f\vec{F}) = f(\nabla \cdot \vec{F}) + (\nabla f) \cdot \vec{F}$
where $\vec{F} = \langle M, N, P \rangle$ and $f: \mathbb{R}^3 \to \mathbb{R}$

$fF = \langle fM, fN, fP \rangle$

$\nabla \cdot f\vec{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle fM, fN, fP \rangle = \partial_x(fM) + \partial_y(fN) + \partial_z(fP)$

$= f_x M + f M_x + f_y N + f N_y + f_z P + f P_z =$

$= f(M_x + N_y + P_z) + (f_x M + f_y N + f_z P) =$

$= f(\nabla \cdot \vec{F}) + \langle f_x, f_y, f_z \rangle \cdot \langle M, N, P \rangle =$

$= f(\nabla \cdot \vec{F}) + \nabla f \cdot \vec{F}$

q.e.d.

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Panel 3

Ex: Take $f(x,y,z) = x^2y^2 - xz$ and

$\vec{F}(x,y,z) = \langle x^2, xz, yz^2 \rangle$

Find the quantities that make sense:

$\nabla f = \langle 2xy^2 - z, 2x^2y, -x \rangle$ $\qquad \cancel{\nabla F}$

$\nabla \cdot F = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle x^2, xz, yz^2 \rangle = 2x + 0 + 2yz$ $\qquad \cancel{\nabla f}$

div(F)

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & xz & yz^2 \end{vmatrix} = \langle z^2 - x, -0, z - 0 \rangle = \underline{\langle z^2 - x, 0, z \rangle}$$

curl(F)

$\cancel{F \times f}$

$fF = \langle (x^2y^2 - xy)\, x^2, (\quad)\, xz, (\quad)\, yz^2 \rangle$

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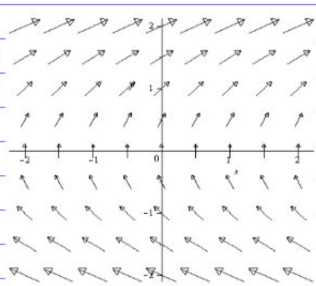
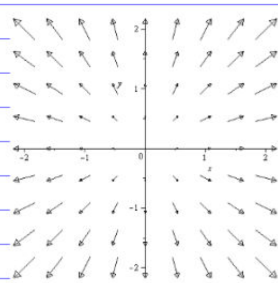
Panel 4

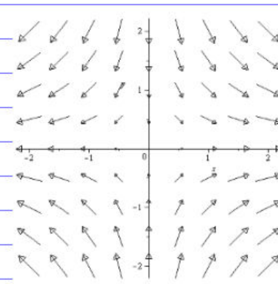
Name: ______

Quiz

① Match the algebraic expressions with the fieldplots:

(i) 

(ii) 

(iii) 

A) $F(x,y) = \langle x, y \rangle$

C) $F(x,y) = \langle y, 1 \rangle$

B) $F(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2+4}}, \frac{y}{\sqrt{x^2+y^2+4}} \right\rangle$

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Panel 5

(2) Suppose $F(x,y,z) = \langle z^2 - x^2,\ 2xy,\ xy+3 \rangle$. Find

a) $\operatorname{div}(F)$

b) $\operatorname{curl}(F)$

(3) Find the vector field with potential $f(x,y,z) = xze^{y^2}$

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Panel 6

<u>Thm:</u> Suppose $\vec{F} = \langle M, N, P \rangle$ is a 3D vector field. Then if F is conservative then $\operatorname{curl}(F) = 0$

$$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \qquad 0$$

<u>Thm:</u> If $\vec{F} = \langle M, N \rangle$ is a 2D conservative vector field. Then $N_x = M_y$

$$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} = \langle 0 - N_z, -(-M_z), N_x - M_y \rangle = \langle 0, 0, N_x - M_y \rangle = \vec{0}$$

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Panel 7

Which of the following vector fields is not conservative

(a) $F(x,y) = \langle x, y \rangle$ ✓

(b) $F(x,y) = \langle x^2+y^2, 2xy \rangle$ ✓

(c) $F(x,y) = \langle e^x \cos(y), -e^x \sin(y) \rangle$ ✓

(d) $F(x,y) = \langle x^2 \cos(y), -y^2 \sin(x) \rangle$ ✗ not conservative!

$N_x = M_y$

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Panel 8

Find potential function for $\vec{F} = (\overset{M}{3+2xy}, \overset{N}{x^2-3y^2})$ if exists

"antiderivative" for vector fields

Since $M_x = M_y = 2x$ there is f s.t. $(f_x, f_y) = \vec{F}$

$f_x = 3+2xy \Rightarrow f = \int f_x \, dx = \int 3+2xy \, dx = 3x + x^2y + C(y)$

$f = 3x + x^2y + C(y)$

$f_y = \cancel{x^2} + C'(y) = \cancel{x^2} - 3y^2 \Rightarrow C'(y) = -3y^2 \Rightarrow C(y) = -y^3 + d$

$\Rightarrow f(x,y) = 3x + x^2y - y^3 + d$

Check: $(f_x, f_y) = (3+2xy, x^2-3y^2) = \vec{F}$

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Panel 9

Find potential function for $\langle x^2\cos(y), -y^2\sin(x)\rangle$

① $f_x = M$: $f_x = x^2\cos(y)$

② integrate w.r.t. x: $f = \frac{1}{3}x^3\cos(y) + C(y)$

③ Find ∂_y : $f_y = -\frac{1}{3}x^3\sin(y) + C'(y) = -y^2\sin(x)$

④ Set equal to M, solve.

x's do not drop out!

there is no potential function

FIRST: check $N_x = M_y$

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Panel 10

Find potential for $\vec{F} = \langle y^2, 2xy + e^{3z}, 3ye^{3z}\rangle$ if exists.

To check conservative: curl $(F) = 0$ → too much!

Want f s.t. $(f_x, f_y, f_z) = \vec{F}$

→ $f_x = y^2$ → $f = xy^2 + C(y,z)$

$f_y = 2\cancel{x}y + C_y(y,z) = 2\cancel{x}y + e^{3z}$ ⇒ $C_y = e^{3z}$ ⇒ $C = ye^{3z} + d(z)$

$f = xy^2 + ye^{3z} + d(z)$

$f_z = \cancel{3ye^{3z}} + d'(z) = \cancel{3ye^{3z}}$ → $d'(z) = 0$ ⇒ d is const.

⇒ $f(x,y,z) = xy^2 + ye^{3z} + d$

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Panel 11

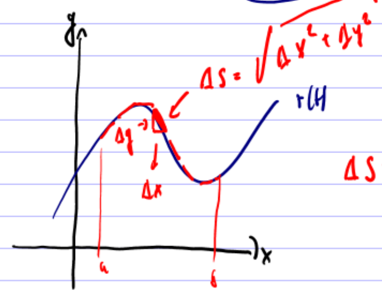
Line Integrals

space curves

Suppose $r(t) = \langle g(t), h(t) \rangle$ describes a curve C in $\mathbb{R}^2$ and $f(x,y)$ is a function defined on C. Then we define the line integral of f along C as:

length of curve

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x')^2+(y')^2}\,dt$$



$$\Delta s = \sqrt{\frac{\Delta x^2}{\Delta t^2} + \frac{\Delta y^2}{\Delta t^2}}\,\Delta t = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$$

$$\lim_{t_1 \to t_2} \frac{x(t_1) - x(t_2)}{t_1 - t_2} = x'(t)$$

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Panel 12

Ex: Find $\int x y^2 ds$, where $\vec{r}$ describes first quarter circle, radius 1.

First: what's the curve: next line

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