Panel 1

Birds-Eye View so far

function $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$ Calc 1/2

$f: \mathbb{R} \to \mathbb{R}^2$ $r(t) = (\sin(t), \cos(t))$ space curves

function $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = \sin(x)\cos(y)$ surfaces

function $f: \mathbb{R}^3 \to \mathbb{R}$ skipped this in terms of integration

Next: $f: \mathbb{R}^2 \to \mathbb{R}^2$ vector fields

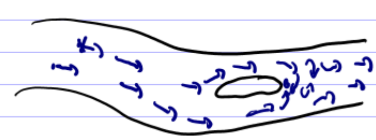
$f(\text{vector}) \to \text{vector}$

1

Panel 2

Vector Fields: If for each point P in a region R there is a unique vector having initial point P, then the totality of such vectors is called a vector field

Ex: Flow of water



Ex: Magnetic Field

Ex: Gravity Fields

2

Panel 3

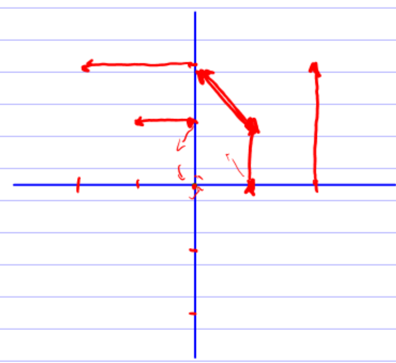
Mathematically, a vector field is given as:

$F(x,y) = \langle xy, x^2+y^2 \rangle \quad : \mathbb{R}^2 \to \mathbb{R}^2 \quad , F(x,y) = \langle xy, x, x^2+y^2 \rangle$

$F(x,y,z) = \langle xy, yz, xz+y+x \rangle : \mathbb{R}^3 \to \mathbb{R}^3 \quad , F(x,y,z) = \langle xyz, x^2 y \rangle$

Ex: Describe $F(x,y) = \langle -y, x \rangle = -y\vec{i} + x\vec{j}$

(x,y)	$F(x,y)$
$(0,0)$	$\langle 0,0 \rangle$
$(1,0)$	$\langle 0,1 \rangle$
$(2,0)$	$\langle 0,2 \rangle$
$(0,1)$	$\langle -1,0 \rangle$
$(0,2)$	$\langle -2,0 \rangle$
$(1,1)$	$\langle -1,1 \rangle$
$\vdots$	



3

Panel 4

Def: If $r(x,y,z) = \langle x,y,z \rangle$ then $F(x,y,z) = \frac{c}{\|r\|^2} \vec{u}$ where $u = r/\|r\|$ is called inverse square field.

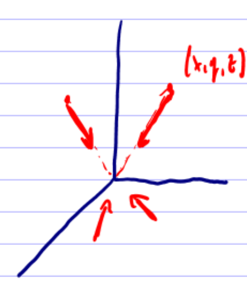
Ex: Describe inverse square field for $c = -1$.

$$F(x,y,z) = \frac{-1}{\|r\|^2} \vec{r} = \underbrace{\frac{-1}{(x^2+y^2+z^2)^{3/2}}} \langle x,y,z \rangle = \underbrace{\frac{1}{(x^2+y^2+z^2)^{3/2}}}_{\text{length}} \underbrace{\langle -x,-y,-z \rangle}_{\text{direction}}$$

Gravity vector field

a) At each point vector points towards origin

b) The closer to origin you get, the longer the vector!



4

Panel 5

Maple offers "fieldplot" and "fieldplot 3d"

> *with*(*plots*);

> *fieldplot*([−y, x], x = −2..2, y = −2..2);

> $fieldplot3d\left(\left[\frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right], x=-2..2, y=-2..2, z=-2..2\right);$

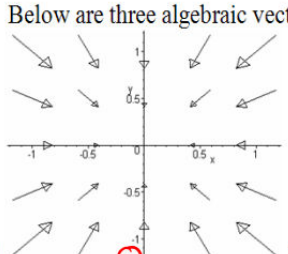
>

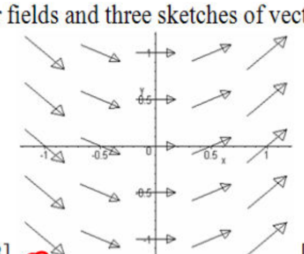
5

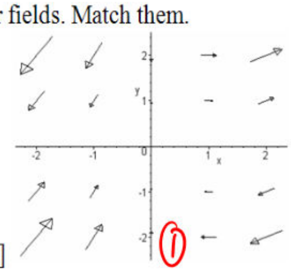
Panel 6

Quiz 8 - Part 1 Friday

1. Below are three algebraic vector fields and three sketches of vector fields. Match them.

[A]

3

[B]

2

[C]

1

(1) $F(x,y) = < xy, y(x-1) >$

(2) $F(x,y) = < 1, x >$

(3) $F(x,y) = < -x, -y >$

6

Panel 7

<u>Def:</u> Suppose $F(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$

Then

$$\operatorname{curl}(F) = \langle P_y - N_z, -(P_x - M_z), N_x - M_y \rangle \leftarrow \text{vector field}$$

$$\operatorname{div}(F) = M_x + N_y + P_z \qquad \leftarrow \underline{\text{function}}$$

Ex. $F(x,y,z) = \langle \overset{M}{xy}, \overset{N}{yz}, \overset{P}{xz} \rangle$

div(curl(F))

curl(~~div(F)~~)

$$\operatorname{div}(F) = y + z + x$$

$$\operatorname{curl}(F) = \langle 0 - y, -(z - 0), 0 - x \rangle = \langle -y, -z, -x \rangle$$

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Panel 8

<u>How to remember:</u>

$$\operatorname{curl}(F) = \nabla \times F =$$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \langle P_y - N_z, -(P_x - M_z), N_x - M_y \rangle$$

$$\operatorname{div}(F) = \nabla \underset{\text{dot}}{\cdot} F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle M, N, P \rangle = \frac{\partial}{\partial x} M + \frac{\partial}{\partial y} N + \frac{\partial}{\partial z} P$$

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Panel 9

Ex! $F(x,y,z) = \langle xy^2z^4,\ 2x^2y + z,\ y^3z^2 \rangle$

Find curl (F) and div (F)

$\operatorname{div}(F) = \nabla \cdot F = y^2z^4 + 2x^2 + 2y^3z$

$$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy^2z^4 & 2x^2y+z & y^3z^2 \end{vmatrix} =$$

$$\langle 3y^2z^2 - 1,\ -(0 - 4xy^2z^3),\ 4xy - 2xyz^4 \rangle$$

9

Panel 10

2. Suppose that $F(x,y,z) = \langle x^3z, x^2z, xy \rangle$ is some vector field.

a) Find $div(F)$

$3x^2z + 0 + 0 = 3x^2z$

b) Find $curl(F)$

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Panel 11

Def: A vector field $\vec{F}$ (M,N,P) is conservative if

there is a function f s.t. $\nabla f = F$, i.e. $f_x = M$, $f_y = N$, $f_z = P$

The function f is called the Potential Function of F.

Ex: Find vector field with potential

$f(x,y,z) = x^2 - 3y^2 + 4z^2$

"anti-derivative" of vector field

$\nabla f = \langle 2x, -6y, 8z \rangle = \vec{F}(x,y,z)$

F is conservative with potential function f

start with F. find f s.t. $f' = F$. f is called

x $\quad f = \frac{1}{2}x^2$

11

Panel 12

Which of the following vector field(s) has as potential function $f(x,y,z) = x^2y^2z^2 + xy + zy$

(a) $\vec{F} = \langle 2x, y, z \rangle$

(b) $\vec{F} = \langle 2xy^2z^2 + x + y \rangle$

(c) $\vec{F} = \langle 2xy^2z^2, y, z \rangle$

(d) $\vec{F} = \langle 2xy^2z^2, x, y \rangle$

$f_x = 2xy^2z^2 + y$

$f_y = 2x^2yz^2 + x + z$

$f_z = 2x^2y^2z + y$

none of these

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Panel 13

Suppose $\vec{F}$ is conservative, i.e. $\nabla f = F$, i.e.

$\vec{F} = \langle f_x, f_y, f_z \rangle = \langle M, N, P \rangle$

$$\text{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle = \langle 0, 0, 0 \rangle$$

Theorem: If a 3D vector field is conservative, and all component functions are twice cont. diff'ble, then

$$\text{curl}(F) = 0$$

picture

(∇f)

$\nabla \times F$

$\nabla \cdot F$

Examples: HW

13

Panel 14

Which of the following vector fields is not conservative

(a) $F(x,y) = \langle x, y \rangle$

(b) $F(x,y) = \langle x^2 + y^2, 2xy \rangle$

(c) $F(x,y) = \langle e^x \cos(y), -e^x \sin(y) \rangle$

(d) $F(x,y) = \langle x^2 \cos(y), -y^2 \sin(x) \rangle$

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