Panel 1

Last Time

Integration:

(1) $\iint_R f(x,y)\,dA = \lim \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i,y_j)\,dA_{ij}$, $\Delta x_i \Delta y_j$

(2) Geometry: volume under surface $z = f(x,y)$ (≥ 0)

(3) How-to: (Fubini) $\int_a^b\int_c^d f\,\underline{dx\,dy} = \int_c^d\int_a^b f\,\underline{dy\,dx}$

same: $\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

1

Panel 2

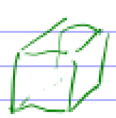
Tripple Integration

Just like partial derivatives, integration can be extended to higher dimension without problems.

Ex: $\iiint_R xyz\,dV$, R a cube in 3D, side length 2

$\underset{x}{[0,2]} \times \underset{y}{[0,2]} \times \underset{z}{[0,2]}$

$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = \int_0^2\int_0^2 \left[\frac{1}{2}x^2yz\Big|_{x=0}^{x=2}\right] dy\,dz = \int_0^2\int_0^2 2yz\,dy\,dz =$



$= \int_0^2 \left[y^2 z\Big|_{y=0}^{2}\right] dz = \int_0^2 4z\,dz = 2z^2\Big|_0^2 = \underline{8}$ m

2

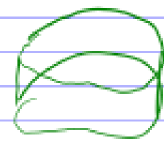
Panel 3

Properties of Double Integrals

(1) $\iint_D f(x,y) + g(x,y)\, dA = \iint_D f\, dA + \iint_D g\, dA$

(2) If $f(x,y) \geq g(x,y)$ then $\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$

(3) $\iint_D 1\, dA = 1 \cdot \text{area}(D) = \text{area}(D)$



D

(4) If $m \leq f(x,y) \leq M$ then

$$\iint_D m\, dA \leq \iint_D f(x,y)\, dA \leq \iint_D M\, dA$$

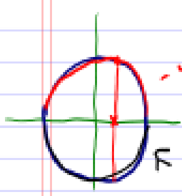
$$m \cdot \text{area}(D) \leq \ldots \leq M\, \text{area}(D)$$

3

Panel 4

Ex: Estimate $\iint_D e^{\sin(x)\cos(y)}\, dA$ where D is disk, radius 2

Setup: $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} e^{\sin(x)\cos(y)}\, dy\, dx$ too much!



$y = \sqrt{4-x^2}$

$y = -\sqrt{4-x^2}$

$x^2+y^2=4$

$-1 \leq \sin(x)\cos(y) \leq 1$ $\quad |e^{(\,)}|$

$\frac{1}{e} \leq e^{\sin(x)\cos(y)} \leq e$

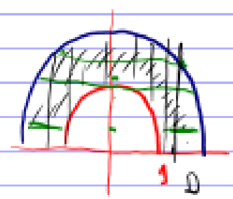
$\Rightarrow \quad 4\pi\frac{1}{e} \leq \iint_D e^{\sin(x)\cos(y)}\, dA \leq e\, 4\pi$

4

Panel 5

But there are some integral where all tricks (so far) don't work:

$$\iint_D (3x^2+3y^2)\,dA \quad \text{where } D \text{ is region in upper half plane bounded by } x^2+y^2=1 \text{ and } x^2+y^2=4$$



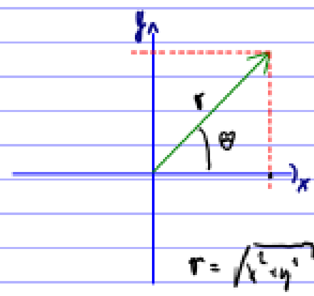
① $\iint \_\_ \,dx\,dy$ 3 integrals

② $\iint \_\_ \,dy\,dx$ 3 integrals

5

Panel 6

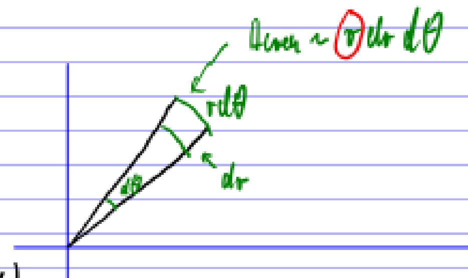
Solution. Polar Coordinates



$x = r\cos(\theta)$

$y = r\sin(\theta)$

$r = \sqrt{x^2+y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$



Thm: $\displaystyle\iint_D f(x,y)\,dA = \iint f(r,\theta)\,r\,dr\,d\theta$

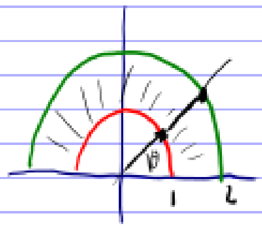
$x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$

Change of variables thm

6

Panel 7

$\iint_D 3x^2+3y^2\,dA$, where D is the region in the upper half plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$



$$\iint_D 3x^2+3y^2\,dA = \int_0^\pi\int_1^2 3(\quad)^2+3(\quad)^2\,r\,dr\,d\theta =$$

$$= \int_0^\pi\int_1^2 \underline{3\,r^2\cos^2(\theta)+3r^2\sin^2(\theta))}\,r\,dr\,d\theta =$$

$x = r\cos\theta$

$y = r\sin\theta$

$$= \int_0^\pi\int_1^2 3r^2\cdot r\,dr\,d\theta = \int_0^\pi\int_0^2 3r^3\,dr\,d\theta =$$

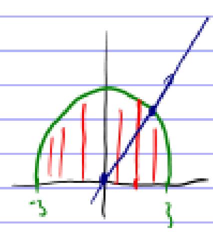
$$= \int_0^\pi \frac{3}{4}\left(r^4\right)\Big|_{r=1}^{2}\,d\theta = \pi\,\frac{3}{4}(16-1) = \underline{\underline{\frac{45}{4}\pi}}$$

7

Panel 8

Ex: $$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^\pi\int_0^3 r \quad r\,dr\,d\theta =$$

$$\left(\int\sqrt{1+y^2}\,dy,\ y=\tan(u)\right)$$



$x = r\cos(\theta)$

$y = r\sin(\theta)$

$$= \int_0^\pi \frac{1}{3}r^3\Big|_0^3\,d\theta = \underline{\underline{9\pi}}$$

$0 \le y \le \sqrt{9-x^2}$

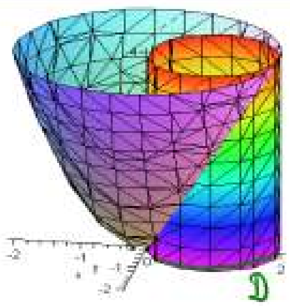
$y = \sqrt{9-x^2}$

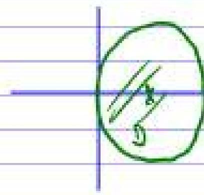
$9 = x^2+y^2$

8

Panel 9

Ex: Volume under $z = x^2+y^2$, inside $x^2+y^2 = 2x$, above xy-plane





must draw $x^2+y^2 = 2x$

$x^2 - 2x + y^2 = 0 \quad |+1$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

with(plots) :

implicitplot3d($\{z = x^2 + y^2, x^2 + y^2 = 2x\}$, $x = -2..2$, $y = -2..2$, $z = 0..4$);

Think "round", i.e. $x = r\cos\theta$, $y = r\sin\theta$:

D: $x^2+y^2 = 2x$

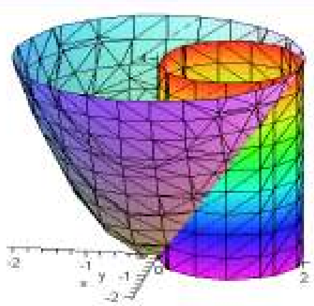
$r^2 = 2r\cos\theta$

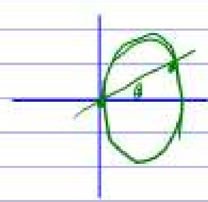
$r = 2\cos\theta$

9

Panel 10

Ex: Volume under $z = x^2+y^2$, inside $x^2+y^2 = 2x$, above xy-plane





$r = 2\cos\theta$

$$\iint_D x^2+y^2 \, dA = \int_0^{\pi} \int_0^{2\cos\theta} r^2 \, r\,dr\,d\theta = \int_0^{\pi} \frac{1}{4} r^4 \Big|_{r=0}^{r=2\cos\theta} d\theta = \int_0^{\pi} 4\cos^4(\theta)\, d\theta = \; =$$

$x = r\cos\theta$

$y = r\sin\theta$

use maple:

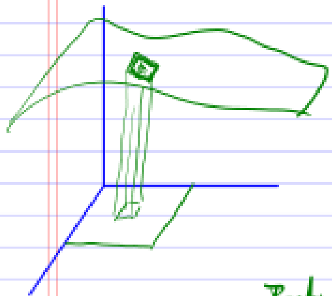
$\frac{1+\cos(2\theta)}{2} = \cos^2\theta$

use this formula 2x

10

Panel 11

Surface Area:



interested in little blotch's area ~ ΔT_{ij}

$A(S) \approx \lim_{n,m\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} dT_{ij}$ is surface area of $z = f(x,y)$ over region D

Fact: $dT_{ij} = \sqrt{f_x^2 + f_y^2 + 1}$ because $(f_x, f_y, -1)$ defines the tangent plane.

Def: Surface area of $z = f(x,y)$ over region D is defined as

$$S = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

Recall: in $\mathbb{R}$ $\int \sqrt{1+(f')^2}\, dx$ was the length of the curve

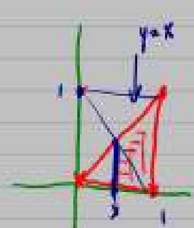
11

Panel 12

Ex: Surface area of $z = x^2 + 2y$ above triangle $(0,0)$, $(1,0)$, and $(1,1)$

$$S = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (2)^2}\, dA =$$

$$= \iint_D \sqrt{4x^2+5}\, dA$$



$$= \int_0^1 \int_0^x \sqrt{4x^2+5}\, dy\, dx =$$

$$= \int_0^1 y\sqrt{4x^2+5}\,\Big|_{y=0}^{y=x} dx = \int_0^1 x\sqrt{4x^2+5}\, dx$$

easy HW

12