Panel 1

Last Time: How to find Extrema $z = f(x,y)$

Relative Extrema	Abs. Extrema
$\nabla f = 0$ (critical points)	$\nabla f = 0$ (critical points)
$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$, $D = f_{xx} f_{yy} - f_{xy}^2$	critical points on bdry
	corner points
Decide	evaluate f at all those points
$D > 0$, $f_{xx} > 0$ min	
$D > 0$, $f_{xx} < 0$ max	
on test — $D < 0$ saddle	~~Lagrange Multipliers~~
Quiz — $D =$ no info	

Exam

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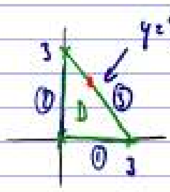
Panel 2

Ex: Let $f(x,y) = 3xy - 6x - 3y + 7$. Find abs. extrema over triangle with corners $(0,0)$, $(3,0)$, and $(0,3)$



D is closed + bdd, f cont. $\Rightarrow$ there is abs max/min

$\nabla f = 0$: $f_x = 3y - 6 = 0 \Rightarrow y = 2$

$f_y = 3x - 3 = 0 \Rightarrow x = 1$

(x,y)	$f(x,y)$
$(1,2)$	[illegible]
$(0,0)$	7
$(3,0)$	[illegible]
$(0,3)$	[illegible]

① $y = 0$: $f(x) = -6x + 7$ no criticals

② $x = 0$: $f(y) = -3y + 7$ no criticals

③ $y = 3 - x$: $f(x) = 3x(3-x) - 6x - 3(3-x) + 7 =$

$= 9x - 3x^2 - 6x - 9 + 3x + 7 = -3x^2 + 6x - 2$, $f'(x) = -6x + 6 = 0 \Rightarrow x = 1$, $y = 2$

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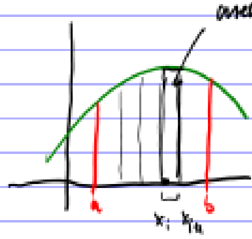
Panel 3

Integration: in $\mathbb{R}$: $f: \mathbb{R} \to \mathbb{R}$

area $= f(x_i)\,\Delta x$



How to do it (2nd Fund. Thm of Calc)

Def: $\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = F(b) - F(a)$

geometrically: area under curve, if $f \geq 0$

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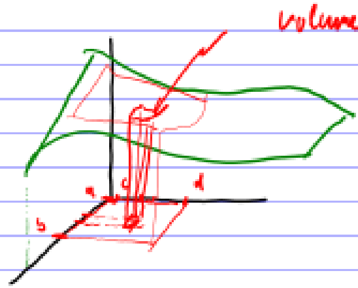
Panel 4

Integration in $\mathbb{R}^2$

$\Delta x \cdot \Delta y$

volume of this column is $f(x_i, y_i)\,dA$



Def: $\int_a^b \int_c^d f(x,y)\,dA = \lim_{n,m \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j)\,\Delta x_i\,\Delta y_j$

Represents geometrically volume under $f(x,y)$ over $[a,b] \times [c,d]$, if $f(x,y) \geq 0$

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Panel 5

Fubini's Theorem (How to integrate in $\mathbb{R}^2$)

If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

double-integral

Ex: $\iint_R (x-3y^2)\,dA$, $R=[0,2]\times[1,2]$

$$\int_1^2\int_0^2 x-3y^2\,dx\,dy = \int_1^2\left(\int_0^2 x-3y^2\,dx\right)dy = \int_1^2 \tfrac{1}{2}x^2-3xy^2\Big|_{x=0}^{x=2}\,dy =$$

$$= \int_1^2 2-6y^2-0\,dy = 2y-2y^3\Big|_1^2 = (4-16)-(2-3)$$

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Panel 6

Similarly: $\int_0^2\int_1^2 x-3y^2\,dy\,dx$ = same

confirm as HW

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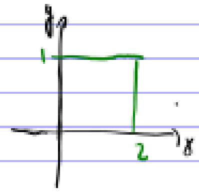
Panel 7

Ex: Find the volume of the solid bounded by $x^2+y^2+z=16$, the planes $x=2$ and $y=1$, and the coordinate planes.



$z=f(x,y)=16-x^2-y^2$ ✓ paraboloid going down

$$V=\int_0^1\int_0^2 16-x^2-y^2\;dx\,dy=$$

$$=\int_0^1\left[16x-\frac{1}{3}x^3-xy^2\Big|_0^2\right]dy$$ = continue.

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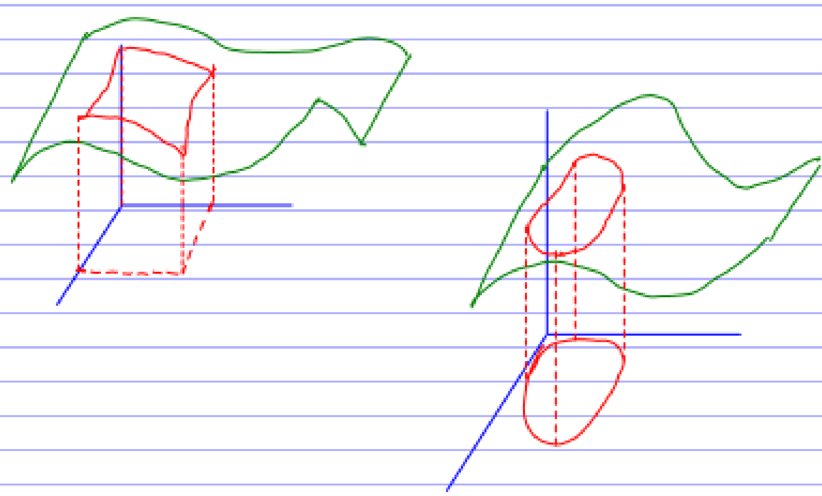
Panel 8

In $\mathbb{R}$ all we ever did was integrate over intervals $[a,b]$. In $\mathbb{R}^2$ it is different:



i.e. more fun in $\mathbb{R}^2$

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Panel 9

Type 1 Region: $D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ $x \in$ interval

Type 2 Region: $E = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$ $y \in$ interval

For type 1:

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

For type 2:

$$\iint_E f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

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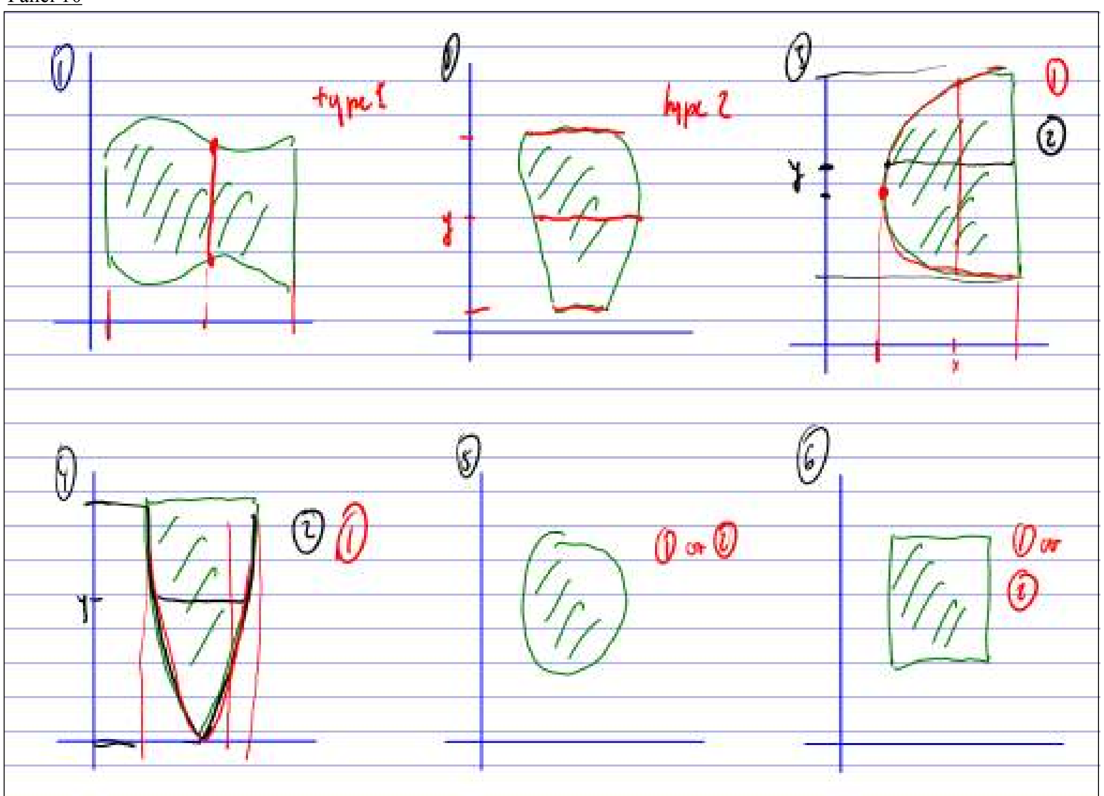
Panel 10



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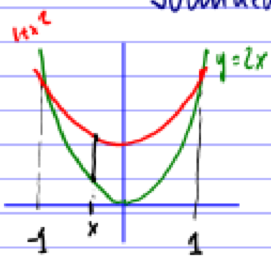
Panel 11

Ex: Find $\iint_D (x+2y)\,dA$ where D is the region bounded by $y=2x^2$ and $y=1+x^2$



type I:

$$\iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx =$$

$$= \int_{-1}^{1}\left[\,xy+y^2\Big|_{y=2x^2}^{y=1+x^2}\right]dx =$$

$$= \int_{-1}^{1} x(1+x^2)+(1+x^2)^2-\left[x\,2x^2+(2x^2)^2\right]dx$$

$= \ldots$ HW

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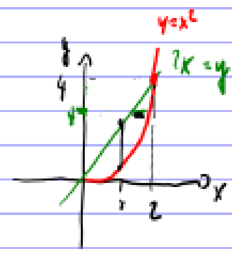
Panel 12

Ex: Volume under $z=x^2+y^2$ above $y=2x$ and $y=x^2$



A $\displaystyle\int_0^4\int_{\cancel{2x}\ \frac{1}{2}y}^{\cancel{x^2}\ \sqrt{y}} x^2+y^2\,dx\,dy$

B $\displaystyle\int_0^2\int_{x^2}^{2x} x^2+y^2\,dy\,dx$

comes out the same

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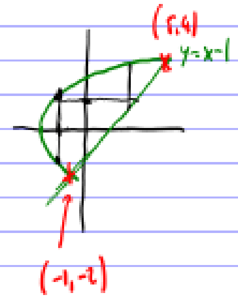
Panel 13

Ex: Find $\iint_D xy \, dA$ where D is bounded by $y = x-1$ and $y^2 = 2x+6$. Should you $\iint xy \, dx\,dy$ or $\iint xy \, dy\,dx$?



$$\int \int_{h(y)}^{h_2(y)} f(x,y)\, dx\,dy$$ 1 integral only

$$\iint f(x,y)\, dy\,dx$$ 2-integrals

$y^2 = 2x+6$ $\quad$ $x-1 = y$

$y^2 - 6 = 2x$ $\quad$ $x = y+1$ $\quad \Rightarrow \frac{1}{2}y^2 - 3 = y+1$ solve: $y = -2$ or 4

$\frac{1}{2}y^2 - 3 = x$

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