Panel 1

Last Time: How to find Relative Extremas

① Find ∇f $((f_x, f_y))$

② $\nabla f = 0$ 2 equations $\Rightarrow$ critical point

③ $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ and $\underline{D} = f_{xx} \cdot f_{yy} - (f_{xy})^2$

④ $D > 0,\ f_{xx} > 0 \Rightarrow$ min (x^2+y^2)

$D > 0,\ f_{xx} < 0 \Rightarrow$ max $(-x^2-y^2)$

$D < 0, \Rightarrow$ saddle (x^2-y^2)

$D = 0$ no clue

What if $f_{xx} = 0$?

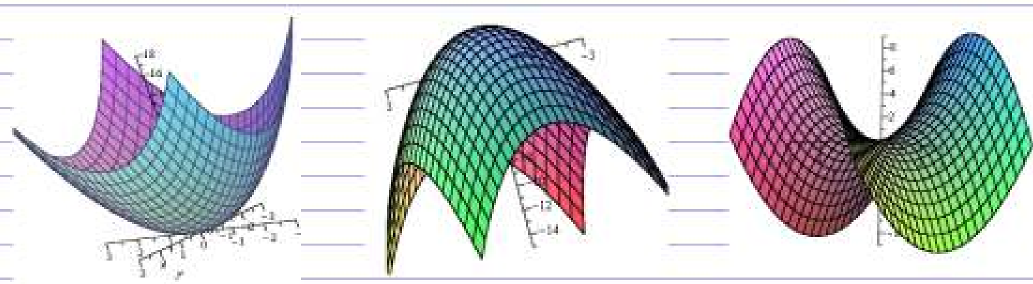
Saddle point or no clue

$D \leq 0$

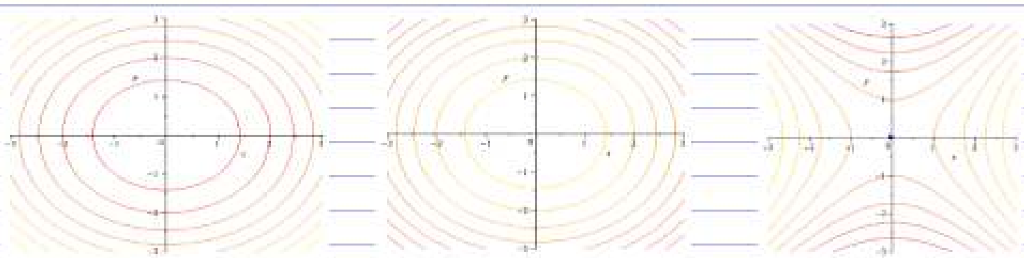
1

Panel 2

Min Max Saddle



$f(x,y) = x^2+y^2$ $f(x,y) = 1-x^2-y^2$ $f(x,y) = x^2-y^2$



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Panel 3

Ex: Find and classify the critical points for $f(x,y) = x^3y + 12x^2 - 8y$

① $f_x = 3x^2y + 24x = 3x(xy+8) = 0 \quad y = -4$ or $\cancel{x=0}$

$f_y = x^3 - 8 = 0 \Rightarrow x = 2$

1 critical point: $(2,-4)$ Saddle point

② $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 3x^2+24 & 3x^2 \\ 3x^2 & 0 \end{pmatrix} \rightarrow D = -9x^4 < 0$ if $x = 2$

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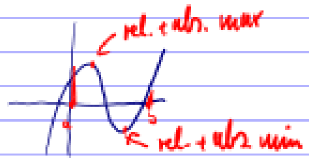
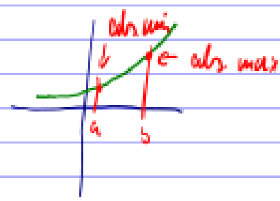
Panel 4

Absolute Max / Min:

Differences between absolute and relative extrema





Thm: If f is continuous on $D \subset \mathbb{R}^2$, D is closed + bounded, then f must have an abs. max and abs. min.

Moreover, they can occur at a critical point or at boundary of D

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Panel 5

To find abs. max/min

If f is continuous on closed bounded set $D \subset \mathbb{R}^2$: e.g. $\overset{x}{[-1,1]} \times \overset{y}{[0,2]}$

1. $Df = 0 \Rightarrow$ critical points
2. Find all critical pts on the boundary of D
3. Add any "corner points"
4. Eval f and pick largest or smallest

Relative

1. $Df = 0$
2. H and D
3. Decide

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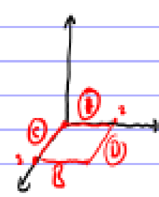
Panel 6

Ex: Find abs. extrema for $f(x,y) = x^2 - 2xy + 2y$ on $[0,3] \times [0,2]$, i.e. $0 \le x \le 3$ and $0 \le y \le 2$

1. $f_x = 2x - 2y = 0 \quad y = 1$ ✓

   $f_y = -2x + 2 = 0 \Rightarrow x = 1$



A) $x = 0$, $y \in [0,2]$: $f(y) = 2y$, $f' = 2 = 0$ none ✓

B) $x = 3$, $y \in [0,2]$: $f(y) = 9 - 6y + 2y \Rightarrow$ none ✓

C) $y = 0$, $x \in [0,3]$: $f(x) = x^2 \Rightarrow x = 0$ critical ✓

D) $y = 2$, $x \in [0,3]$: $f(x) = x^2 - 4x + 4 \Rightarrow x = 2$ critical

(x,y)	$f(x,y) = z$
$(1,1)$	1
$(0,0)$	0 ← abs min
$(2,2)$	0
double ~~$(0,0)$~~	~~0~~
$(3,0)$	9 ← abs. max
$(0,2)$	4
$(3,2)$	2

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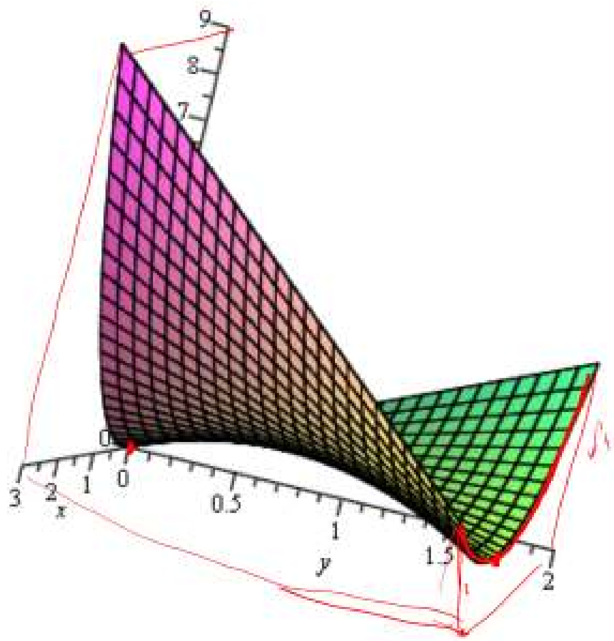
Panel 7



(x,y)	$z=f(x,y)$
$(1,1)$	1
$(2,2)$	0
$(0,0)$	0
$(0,2)$	4
$(3,0)$	9 ✓
$(3,2)$	2

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Panel 8

$f(x,y) = x^2 + 2y^2 + 4xy$ for $(x,y) \in [0,1] \times [0,1]$. → abs. extrema?

① $f_x = 2x + 4y = 0 \Rightarrow x = 0$

$f_y = 4y + 4x = 0 \qquad y = 0$

② $x=0$: $f(y) = 2y^2 \Rightarrow y = 0$

$x = 1$: $f(y) = 1 + 2y^2 + 4y \Rightarrow y = \cancel{-1}$

$y = 0$: $f(x) = x^2 \Rightarrow x = 0.$

$y = 1$: $f(x) = x^2 + 2 + 4x \Rightarrow x = \cancel{-2}$

$2x + 4 = 0$

(x,y)	$f(x,y)$
$(0,0)$	0
$(1,0)$	1
$(0,1)$	2
$(1,1)$	7

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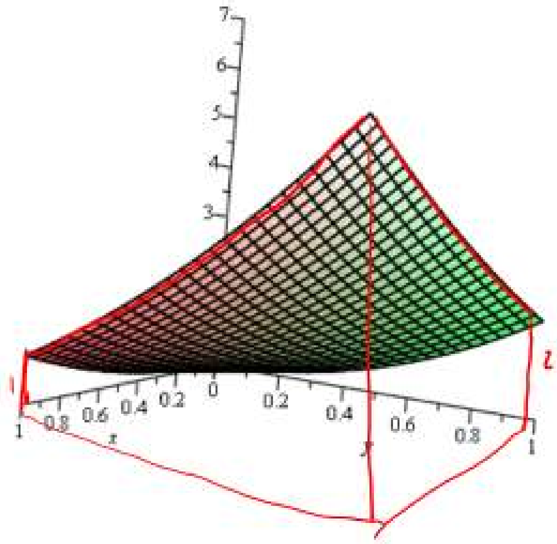
Panel 9



$f(x,y) = x^2 + 2y^2 + 4xy$

(x,y)	$f(x,y) = z$
$(0,0)$	0 ← abs min
$(0,1)$	2
$(1,0)$	1
$(1,1)$	7 ← abs max

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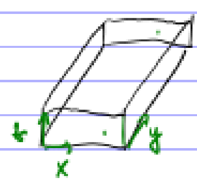
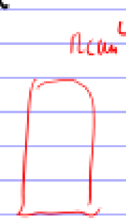
Panel 10

Ex: Make a box w/o lid out of $12\ cm^2$ cardboard of max volume.

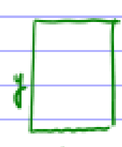


$V = xyz$

$2xz + 2yz + xy = 12$



One Dimension lower: $\mathbb{R}$



$P = 2x + 2y = 12 \iff y = 6 - x$

$A = xy$ max

$A(x) = x(6-x) = 6x - x^2$

$A'(x) = 6 - 2x = 0 \Rightarrow \boxed{x = 3} \Rightarrow \boxed{y = 3}$

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Panel 11

Problem: Max $V = xyz$ subject to $2xz + 2yz + xy = 12$

$2xz + 2yz + xy = 12$

$z(2x+2y) = 12 - xy$

$\Rightarrow z = \dfrac{12-xy}{2x+2y}$

$$V(x,y) = \frac{xy(12-xy)}{2x+2y}$$

Next: $V_x = 0$, $V_y = 0$ $\}$ solve

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Panel 12

Problem: Max $V = xyz$ subject to $2xz + 2yz + xy = 12$

$$\Rightarrow V(x,y) = \frac{xy(12-xy)}{2x+2y}$$

$V(x, y) := \dfrac{x\,y\,(12 - x\,y)}{2\,x + 2\,y}$

$$(x, y) \rightarrow \frac{x\,y\,(12 - x\,y)}{2\,x + 2\,y} \quad (1)$$

$Vx := diff(V(x, y), x)$

$$\frac{y\,(12 - x\,y)}{2\,x + 2\,y} - \frac{x\,y^2}{2\,x + 2\,y} - \frac{2\,x\,y\,(12 - x\,y)}{(2\,x + 2\,y)^2} \quad (2)$$

$Vy := diff(V(x, y), y);$

$$\frac{x\,(12 - x\,y)}{2\,x + 2\,y} - \frac{x^2\,y}{2\,x + 2\,y} - \frac{2\,x\,y\,(12 - x\,y)}{(2\,x + 2\,y)^2} \quad (3)$$

$solve(\{Vx = 0, Vy = 0\}, \{x, y\})$

$$\{x = 2, y = 2\}, \{x = -2, y = -2\} \quad (4)$$

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Panel 13

Other Method: Lagrange Multipliers

Maximize $f(x,y,z)$ subject to $g(x,y,z)=k$

① Solve $\nabla f = \lambda \nabla g$ (grad. f = lambda · grad g)

$g(x,y,z) = k$ 4 X equations, 4 unknowns

② Solve above system. It will give max/min

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Panel 14

Max: $V = xyz$ subject to $2xz + 2yz + xy = 12 = g(x,y,z)$

$\nabla V = \lambda \nabla g$: $yz = \lambda(2z+y) \Rightarrow xyz = \lambda x(2z+y)$ (1)

$xz = \lambda(2z+x)$ $\quad xyz = \lambda y(2z+x)$ (2)

$xy = \lambda(2x+2y)$ $\quad xyz = \lambda z(2x+2y)$ (3)

$2xz + 2yz + xy = 12$ $\quad 2xz + 2yz + xy = 12$ (4)

(1)=(2): $\cancel{\lambda} x(2z+y) = \cancel{\lambda} y(2z+x) \Rightarrow 2x\cancel{z} + \cancel{xy} = 2y\cancel{z} + \cancel{xy} \Rightarrow x = y$

(2)=(3): $\cancel{\lambda}\cancel{y}(2z+y) = \cancel{\lambda} z\, 4\cancel{y} \Rightarrow \boxed{y = 2z = x}$

④ $4z^2 + 4z^2 + 4z^2 = 12 \Rightarrow z = \pm 1 \Rightarrow \boxed{z = 1,\ x = 2,\ y = 2}$

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Panel 15

Integration:

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