Panel 1

Last Time

Chain Rule: $f(x,y)$, $x = x(t)$, $y = y(t)$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Directional Derivative: $D_{\vec{u}}(f) = \nabla f \cdot \vec{u}$, $\vec{u}$ unit vector

Gradient: $\nabla f = \langle f_x, f_y \rangle$

Properties of Gradient: see next slide

1

Panel 2

Properties of Gradient

- The gradient is a vector
- Gradient is perpendicular to level curves
- Gradient points in direction of max. increase
- $\|\nabla f\|$ is the max. rate of change

Ex: Find ∇f if $f(x,y,z) = \ln(xy^2z^3)$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{y^2z^3}{xy^2z^3}, \frac{2xyz^3}{xy^2z^3}, \frac{3xy^2z^2}{xy^2z^3} \right\rangle$$

$$= \left\langle \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \right\rangle$$:-)

2

Panel 3

Ex: Suppose the level curves of an area are given by $f(x,y) = y\ln(x)$. You are standing at $P(1,-3)$ and you are heading in the direction $\langle -4,3\rangle$. Are you going up or down? How much?

Compute $D_{\vec{u}}(f)$ at $P(1,-3)$

$D_{\vec{u}}(f) = \nabla f \cdot \vec{u} = \langle \frac{y}{x}, \ln(x)\rangle \frac{1}{5}\langle -4,3\rangle$ at $P(1,-3)$:

$= \langle -3, 0\rangle \frac{1}{5}\langle -4,3\rangle = \frac{12}{5} > 0$

Thus it's going up by $12/5$ units.

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Panel 4

Name: ________

Quiz

① Consider the function $f(x,y) = x^2 + 3xy - y^2$. Find

a) f_x

b) $\frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \cancel{\frac{\partial f}{\partial y}}\frac{\partial y}{\partial t}$ at $t=0$

c) ∇f

$x(0) = 0$ $y(0) = 1$

② If $f(x,y) = xy + 3xy^2$ and $x = \sin(2t)$, $y = \cos(t)$. Find $\frac{\partial f}{\partial t}$ for $t = 0$:

$\frac{\partial x}{\partial t} = 2\cos(2t) \Rightarrow \frac{\partial x}{\partial t}\Big|_{t=0} = \underline{2}$

$\frac{\partial y}{\partial t} = -\sin(t) \rightarrow \frac{\partial y}{\partial t}\Big|_{t=0} = 0$

$\frac{\partial f}{\partial t} = 4 \cdot 2 + 0 = \underline{\underline{8}}$

$\frac{\partial f}{\partial x} = y + 3y^2$ at $(0,1)$: 4

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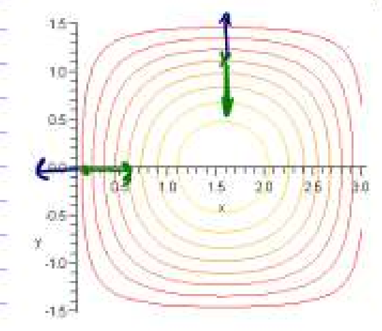
Panel 5

(3) $f(x,y) = x^3 - 3xy + 4y^2$. Find directional derivative in the direction of $\langle \cos(\pi/6), \sin(\pi/6) \rangle = \vec{u}$

$$D_{\vec{u}}(f) = \lim_{h \to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h}, \quad \vec{u} = \langle a, b \rangle$$

$$\nabla f \cdot \vec{u}$$

(4) Consider the contour plot below. Sketch the gradient



a) at $P(1.5, 1.0)$

b) at $P(0,0)$

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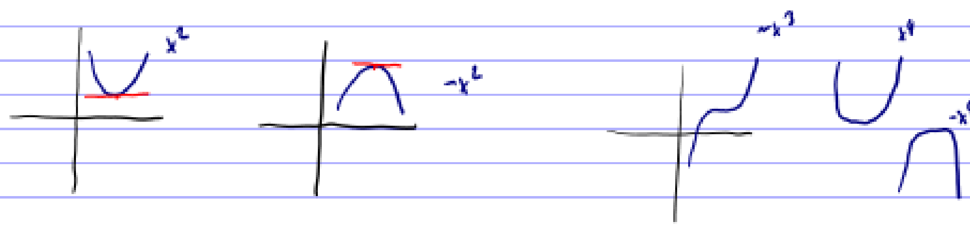
Panel 6

Review of Max/Min problems in $\mathbb{R}$



local min. local max.

$f' = 0$	$f' = 0$	$f' = 0$
$f'' > 0$	$f'' < 0$	$f'' = 0$

(1) f' .

(2) $f' = 0$ (critical)

(3) check f''

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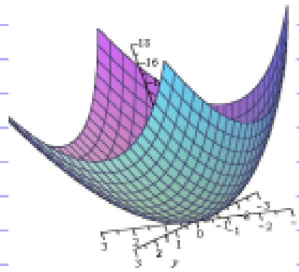
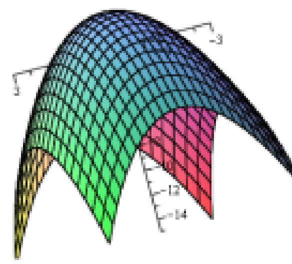
Panel 7

How about in $\mathbb{R}^2$

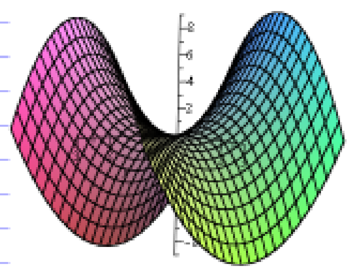
min x^2+y^2



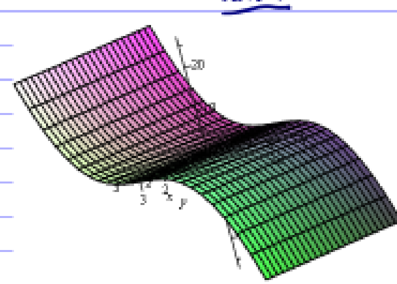
max $-x^2-y^2$



saddle x^2-y^2



neither



neither x^3

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Panel 8

Max / Min Problems

To find max/min of $z = f(x,y)$:

$f_{xy} = f_{yx}$

1. Find Df
2. Solve $Df = 0$ (system of equations)
3. Compute $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ Hessian matrix and $D = f_{xx} f_{yy} - (f_{xy})^2$

Memorize

a) f has min if: $D>0$, $f_{xx}>0$

b) f has max if. $D>0$, $f_{xx}<0$

c) f has saddle if: $D<0$

d) no information if: $D=0$

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Panel 9

Ex: Find and classify the critical points for

$$f(x,y) = x^2 - 2xy + 3y^2 + 4x$$

① ∇f: $f_x = 2x - 2y + 4$

$f_y = -2x + 6y$

② $\nabla f = 0$: $2x - 2y + 4 = 0$

$+ \quad -2x + 6y = 0$

$4y + 4 = 0 \Rightarrow \underline{y = -1,\ x = -3}$ is critical point.

$\boxed{\text{is minimum}}$

③ $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}$: $D = 2\cdot 6 - (-2)(2) = 12 - 4 = 8 > 0$

$f_{xx} = 2 > 0$

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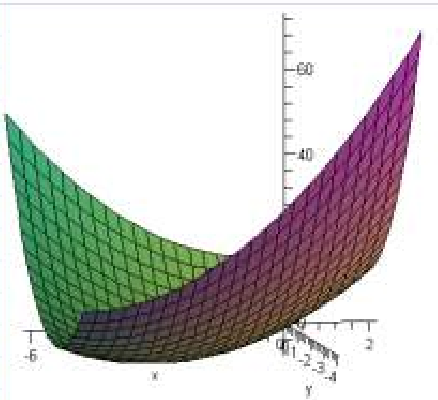
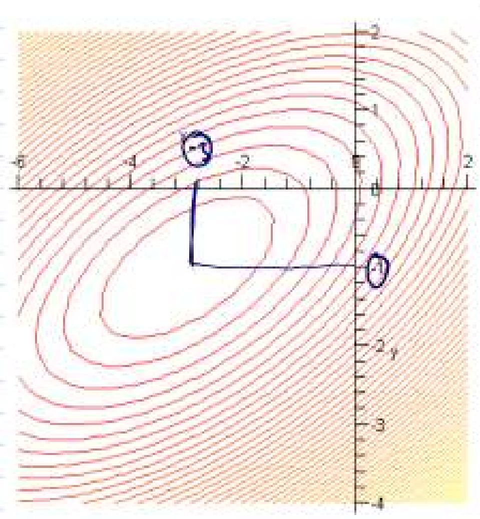
Panel 10

$$f(x,y) = x^2 - 2xy + 3y^2 + 4x$$





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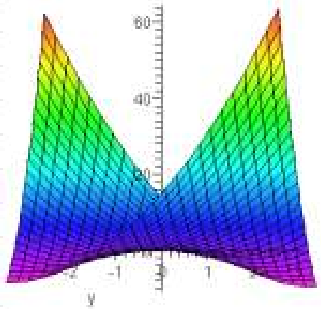
Panel 11

Suppose $f(x,y)=x^2+2y^2+4xy$. Find and classify all relative extrema, if any.



$f_x = 2x + 4y = 0 \quad \Rightarrow 4y = -2x \quad \underline{y=0}$

$f_y = 4y + 4x = 0 \quad \Rightarrow 2x = 0, \underline{x=0}$

$(0,0)$ is critical

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix}$$

$D = 8 - 16 = -8 < 0$

saddle point!

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Panel 12

Find and classify critical points for $f(x,y) = 3x - x^3 - 2y^2$

$f_x = 3 - 3x^2 = 0 \quad \Rightarrow x = 1 \text{ or } -1$ 2 critical points

$f_y = -4y = 0 \quad \Rightarrow y = 0$ $\underline{(1,0)}$ and $\underline{(-1,0)}$

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -6x & 0 \\ 0 & -4 \end{pmatrix} \Rightarrow \boxed{D = 24x}$$

$\underline{\text{Thus:}}$ $(1,0)$: $D > 0$, $f_{xx} < 0 \Rightarrow$ max

$(-1,0)$: $D < 0 \Rightarrow$ saddle

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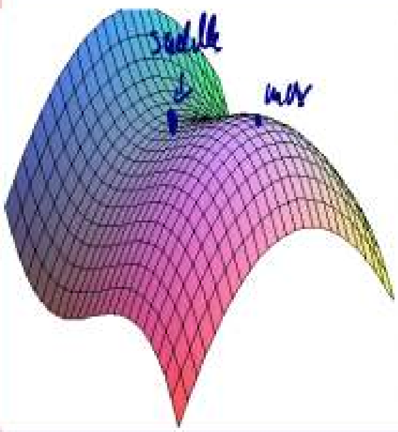
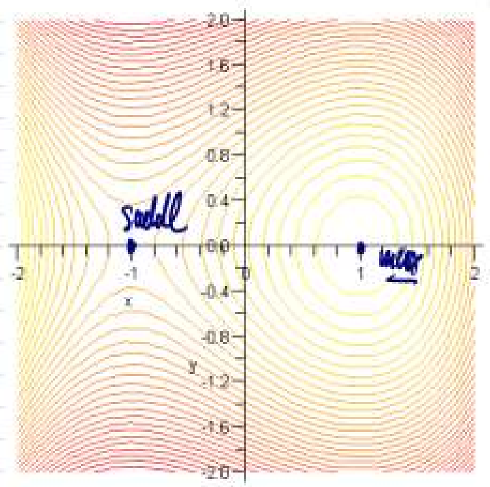
Panel 13

$f(x,y) = 3x - x^3 - 2y^2$





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Panel 14

Ex: Find and classify the critical points for $f(x,y) = x^3 y + 12x^2 - 8y$

Best guess from plot / contourplot is saddle but where?

$\Rightarrow$ HW

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