Panel 1

Pi Day was Yesterday

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Panel 2

Last Time

Partial derivatives : $f_x, f_y, \frac{\partial f}{\partial x}$

Higher - order partials : $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

Tangent planes : plane that "supports" a surface $z = f(x,y)$

Harmonic functions : $f_{xx} + f_{yy} = 0$

Chain Rule : later

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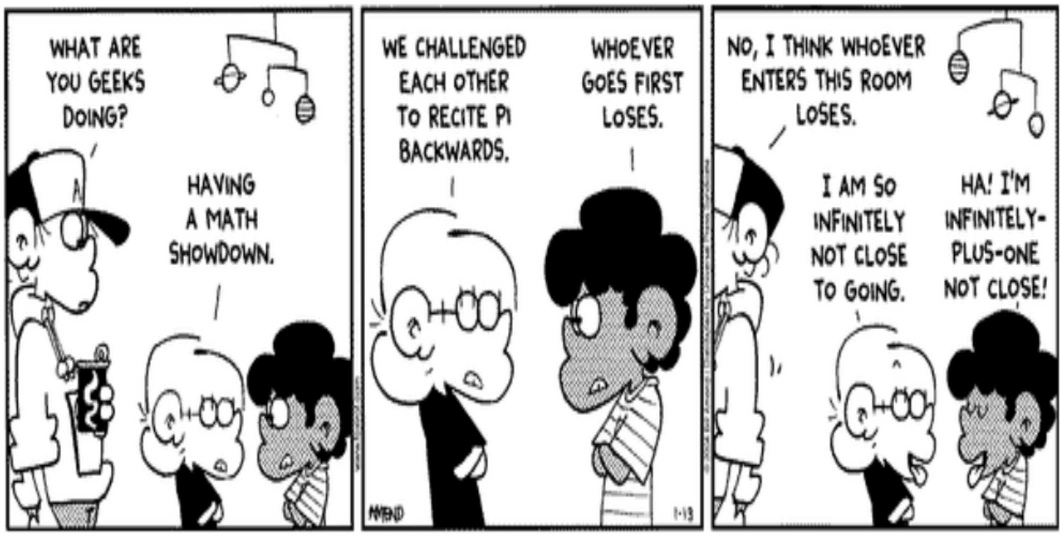
Panel 3



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Panel 4

Ex: Let $f(x, y, z) = xy^2 \cdot \cos(xz)$. Find

$\dfrac{\partial^2 f}{\partial x \partial y} = f_{xy} = (f_x)_y$: $f_x = y^2 \cos(xz) + xy^2(-\sin(xz)) \cdot z$

$(f_x)_y = 2y \cos(xz) + 2y\, x(-\sin(xz))z$

$\dfrac{\partial^3 f}{\partial y^2 \partial z} = f_{yyz}$: $f_y = 2y\, x \cos(xz)$

$f_{yy} = 2x \cos(xz)$

$f_{xxz} = -2x \sin(xz) \cdot x = -2x^2 \sin(xz)$

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Panel 5

Ex: If $f(x,y) = \sin(x)\cosh(y)$ harmonic? $f_{xx} + f_{yy} = 0$

Review: $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$, $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$

$$\frac{d}{dx}\cosh(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$$

$$\frac{d}{dx}\sinh(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$$

$f_x = \cos(x)\cosh(y)$ $\quad f_{xx} = -\sin(x)\cosh(y)$

$f_y = \sin(x)\sinh(y)$ $\quad f_{yy} = \sin(x)\cosh(y)$

$+ \overline{\quad\quad 0 \quad\quad}$

So yes, f is harmonic!

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Panel 6

Equation of tangent plane to $f(x,y)$ at (x_0, y_0) is:

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

Ex: $f(x,y) = x^2 - 3xy + y^2$ when $x=1$ and $y=1$

$f_x = 2x - 3y \Rightarrow f_x(1,1) = -1$

$f_y = -3x + 2y \Rightarrow f_y(1,1) = -1$

$z_0 = f(1,1) = -1$

$$\boxed{z = -(x-1) - (y-1) - 1}$$

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Panel 7

The Chain Rule

$z = f(x,y)$ where $x = x(t)$ and $y = y(t)$

$$\Rightarrow \frac{\partial z}{\partial t} = \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Ex: $f(x,y) = e^{x^2+y^2}$, $x = \cos(t)$, $y = \sin(t)$ $\Big|$ $f(t) = e^{\cos^2(t)+\sin^2(t)} = e^1$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} =$$

$$= 2x\, e^{x^2+y^2}(-\sin(t)) + 2y\, e^{x^2+y^2}\cos(t)$$

$$= -2xy\, e^{x^2+y^2} + 2xy\, e^{x^2+y^2} = 0$$

Panel 8

Ex: Suppose $x^3 + y^3 = 6xy$ defines $y = y(x)$ implicitly

$F(x,y) = 0 \iff \boxed{x^3 + y^3 - 6xy} = 0$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = F_x \cdot 1 + F_y\, y' = 0$$

$$y' = -F_x / F_y$$

$F_x = 3x^2 - 6y$, $F_y = 3y^2 - 6x$ $\Rightarrow y' = -\dfrac{3x^2 - 6y}{3y^2 - 6x}$

$x^3 + y^3 - 6xy = 0 \quad \Big| \frac{\partial}{\partial x}$

$3x^2 + 3y^2 \cdot y' - 6y - 6x\, y' = 0$

$$y' = -\frac{3x^2 - 6y}{3y^2 - 6x}$$

Panel 9

Directional Derivatives

$$f_x = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$ is deriv. in x-direction, or $\langle 1,0\rangle$

$$f_y = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$ is deriv. in y-direction, or $\langle 0,1\rangle$

unit vector

Def: Directional derivative in direction $\vec{u} = \langle a,b\rangle$

$$D_{\vec{u}} f = \lim_{h\to 0} \frac{f(x+ha,\, y+hb)-f(x,y)}{h}$$

$$D_{\langle 1,0\rangle} f = f_x$$

$$D_{\langle 0,1\rangle} f = f_y$$

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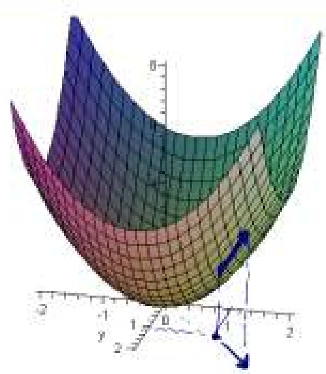
Panel 10

Ex: $f(x,y) = x^2+y^2$. Find directional derivative of f in the direction of $\vec{u} = \frac{1}{\sqrt{2}}\langle 1,1\rangle$ at $(1,1)$:



$$D_{\vec{u}} f = \lim_{h\to 0} \frac{f(x+\frac{h}{\sqrt{2}},\, y+\frac{h}{\sqrt{2}})-f(x,y)}{h} =$$

$$= \lim_{h\to 0} \frac{(x+h/\sqrt{2})^2+(y+h/\sqrt{2})^2-x^2-y^2}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2x\frac{h}{\sqrt{2}}+\frac{h^2}{2}+\cancel{y^2}+2y\frac{h}{\sqrt{2}}+\frac{h^2}{2}-\cancel{x^2}-\cancel{y^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(\sqrt{2}x+\frac{h}{2}+\sqrt{2}y+\frac{h}{2})}{\cancel{h}} = \sqrt{2}x+\sqrt{2}y$$

At $(1,1)$: $D_{\vec{u}} f(1,1) = 2\sqrt{2}$

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Panel 11

Thm: Suppose $\vec{u} = \langle a, b\rangle$ is a unit vector. Then

$$D_{\vec{u}} f(x,y) = f_x \cdot a + f_y \cdot b = \langle f_x, f_y\rangle \cdot \langle a, b\rangle = \langle f_x, f_y\rangle \cdot \vec{u}$$

Ex: $f(x,y) = x^2 + y^2$, $\vec{u} = \frac{1}{\sqrt{2}}\langle 1,1\rangle$. Find $D_{\vec{u}} f(x,y)$ at $(1,1)$

$f_x = 2x$ at $(1,1)$: 2

$f_y = 2y$ at $(1,1)$: 2

$$D_{\vec{u}} f = \langle 2x, 2y\rangle \frac{1}{\sqrt{2}}\langle 1,1\rangle = \frac{1}{\sqrt{2}} 2x + \frac{1}{\sqrt{2}} 2y = \sqrt{2}\,x + \sqrt{2}\,y$$

as before!

Q7 on Wed

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Panel 12

Ex: $f(x,y) = x^3 - 3xy + 4y^2$. Find directional derivative in the direction of $\langle \cos(\pi/6), \sin(\pi/6)\rangle$ ← unit vector?

$$\| \langle \cos(\pi/6), \sin(\pi/6)\rangle \| = \sqrt{\cos^2(\pi/6) + \sin^2(\pi/6)} = 1$$

$f_x = 3x^2 - 3y$

$f_y = -3x + 8y$

$$D_{\vec{u}}(f) = \langle f_x, f_y\rangle \cdot \vec{u} = \langle 3x^2 - 3y, -3x + 8y\rangle \cdot \langle \cos(\pi/6), \sin(\pi/6)\rangle =$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}$$

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Panel 13

<u>Def:</u> If $f(x,y)$ is a function of 2 variables, then the <u>gradient of f</u> is:

$$\text{grad}(f) = \nabla f = \langle f_x, f_y \rangle$$

i.e. the vector with components f_x and f_y

<u>Note:</u> $D_{\vec{u}} f =$

<u>Ex:</u> $f(x,y) = xy\sin(x)$.

$$\nabla f = \langle f_x, f_y \rangle = \langle y\sin(x) + xy\cos(x), x\sin(x) \rangle$$

$$f(x,y,z) = x\cos(y^2+z^2)$$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle \cos(y^2+z^2), -x\sin(y^2+z^2)\cdot 2y, -x\sin(y^2+z^2)2z \rangle$$

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Panel 14

<u>Recall:</u> $D_{\vec{u}} f = \nabla f \cdot u = \|\nabla f\| \cdot \|u\| \cdot \cos(\alpha)$

$$= \|\nabla f\| \cos(\alpha)$$

<u>Theorem:</u> The max. value of $D_{\vec{u}}(f)$ is $\|\nabla f\|$ and is attained if $\vec{u}$ points in direction of $\underline{\nabla f}$

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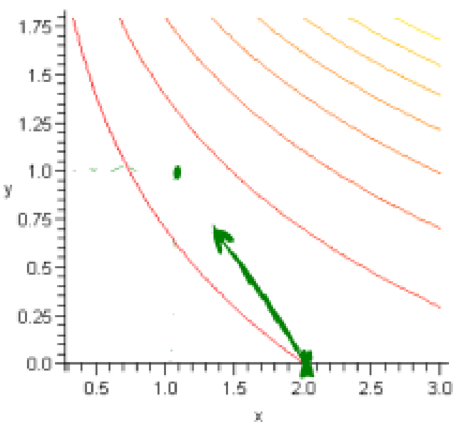
Panel 15

Ex: $f(x,y) = xe^y$. Find rate of change at $P(2,0)$ in the direction from P to $Q(1,1)$.

level curves



$PQ = \langle -1,1\rangle \Rightarrow \vec{u} = \frac{1}{\sqrt{2}}\langle -1,1\rangle$

$D_{\vec{u}} f = \nabla f \cdot \frac{1}{\sqrt{2}}\langle -1,1\rangle =$

$\nabla f = \langle f_x, f_y\rangle = \langle e^y, xe^y\rangle \Big|_{(2,0)}$
$= \langle 1, 2\rangle$

$D_{\vec{u}} f = \langle 1,2\rangle \cdot \frac{1}{\sqrt{2}}\langle -1,1\rangle = \frac{1}{\sqrt{2}}$.

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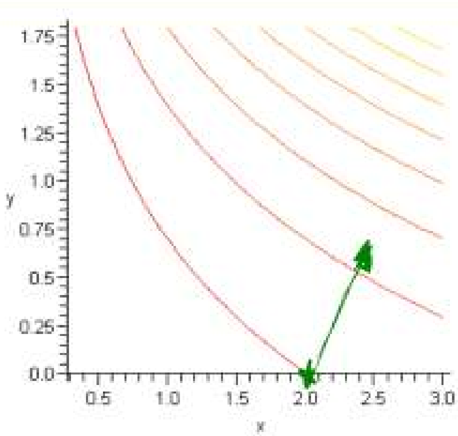
Panel 16

Ex: $f(x,y) = xe^y$. You are standing at $P(2,0)$. In which direction does f change most rapidly, and what is that rate of change?



Answer: $\nabla f = \langle 1,2\rangle$

$\Rightarrow$ the direction of largest rate of change is $\langle 1,2\rangle$ and the largest rate of change is $\|\nabla f\| = \sqrt{5}$

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Panel 17

Summary: If $f(x,y)$ is a function then

$\underline{\nabla} f = (f_x, f_y)$ is the gradient

(a) ∇f is a vector

(b) ∇f is perpendicular to level curves

(c) points in direction of max increase

(d) $\|\nabla f\|$ is largest rate of change.

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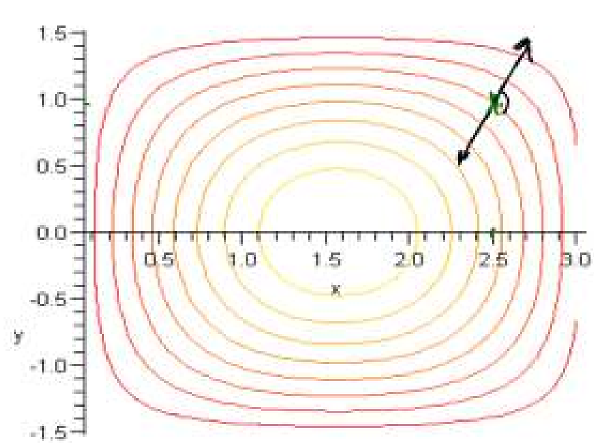
Panel 18

Below is a contour plot for $f(x,y)$, showing several level curves. Sketch ∇f at $P(2.5, 1.0)$, approx.



numeric algorithm to find local max or min! "follow the gradient"

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Panel 19

Ex: Find rate of change of $f(x,y) = y \ln(x)$ at $P(1,-3)$ in the direction $\langle -4/5, 3/5 \rangle$ unit vector?

$D_{\vec{u}} f = \nabla f \cdot \langle -\frac{4}{5}, \frac{3}{5} \rangle = \ldots$

Ex: Find max. rate of change for $f(x,y,z) = \ln(xy^2z^3)$ at $P(1,1,1)$

$\|\nabla f\|$

Goes in the direction ∇f

Have fun with HW

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Panel 20

Review of Max / Min Problems in $\mathbb{R}$:

next time

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