Panel 1

Last time: Review of contour plots and surfaces.

limits $\lim\limits_{(x,y)\to(0,0)} f(x,y)$: $x=0$, $y=0$, $x=y$

continuity $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ $\Leftrightarrow$ continuous at (x_0,y_0)

$f_x(x,y) = \lim\limits_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$ i.e. think y const

$f_y(x,y) = \lim\limits_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$ think x const.

$\frac{\partial^2 f}{\partial x \partial y} = (f_x)_y \overset{!}{=} (f_y)_x$

usually ...

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Panel 2

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

$x=0$: $\lim\limits_{y\to 0} (\quad) = 0$

$y=0$: $\lim\limits_{x\to 0} (\quad) = 1$

$\}$ different, i.e. no limit for

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Panel 3

Ex: Let $f(x,y,z) = xyz\sin(z)$. Find

$$\frac{\partial^2 f}{\partial x \partial y}: \quad f_x = yz\sin(z)$$

$$f_{xy} = z\sin(z)$$

$$xyz\cos(z)$$

$$\frac{\partial^3 f}{\partial z \partial y^2}: \quad f_z = xy\sin(z) + xyz\cos(z)$$

$$\Rightarrow f_{zy} = x\sin(z) + xz\cos(z)$$

$$f_{zyy} = 0$$

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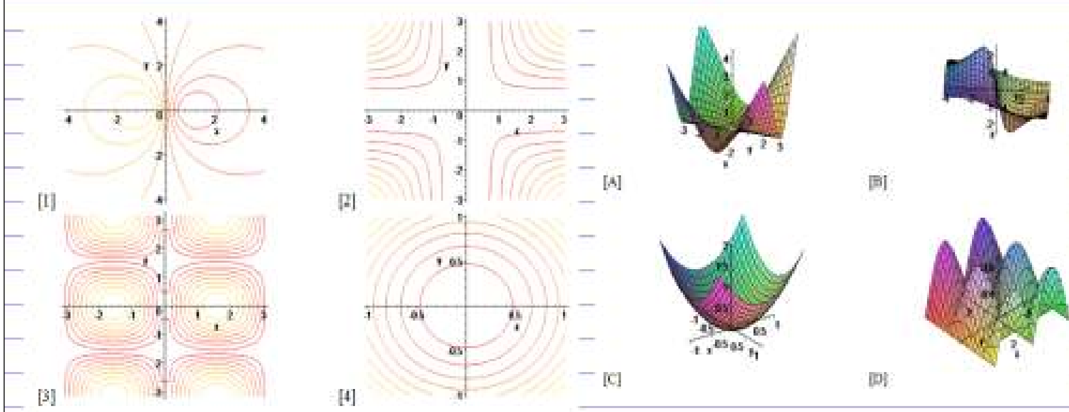
Panel 4

Quiz #5 Name: ____________

① Match the surfaces on the right with the contour plots on the left.



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Panel 5

#2) Find the limit if possible. Justify your argument.

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{4x^2+5y^2}$$

#3) Find the indicated partial derivatives

a) $f(x,y) = xy + x^2y^3$

$f_x(x,y) =$

b) $f(x,y,z) = yz\sin(xz)$

$f_z = \frac{\partial f}{\partial z} = y\sin(xz) + zy \cdot x\cos(xz)$

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Panel 6

Partial derivatives frequently occur in Physics to describe laws of nature as PDEs (partial differential equations). For example: the Laplace PDE

$$\frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f = 0$$

is important in heat conduction and fluid flow.

Ex: Show that $f(x,y) = e^x \sin(y)$ satisfies the above PDE

$f_x = e^x \cdot \sin(y)$ $\quad f_y = e^x \cos(y)$ $\quad$ f satisfies the Laplace PDE

$f_{xx} = e^x \sin(y)$ $\quad f_{yy} = -e^x \sin(y)$

$\Rightarrow f_{xx} + f_{yy} = 0$

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Panel 7

f_x = slope of tangent in x-dir

f_y = slope of tangent in y-direction

Want: Find tangent plane at (x_0, y_0, z_0) to $f(x,y)$

$\Rightarrow$ Find $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ or $ax+by+cz=d$

$z = ax+by+c$

Want: tangent to f = tangent to plane in x-dir, and y-dir.

$\frac{\partial z}{\partial x} = a = f_x$ $\quad \frac{\partial z}{\partial y} = b = f_y$

$$\Rightarrow \boxed{z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0}$$

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Panel 8

Equation of tangent plane to $f(x,y)$ at (x_0,y_0) is:

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

Ex: $f(x,y) = 2x^2+y^2$. Find tangent plane at $P(1,1,3)$

$f_x = 4x$ at P: $f_x = 4$

$f_y = 2y$ at P: $f_x = 2$

$\Rightarrow z = 4(x-1)+2(y-1)+3$

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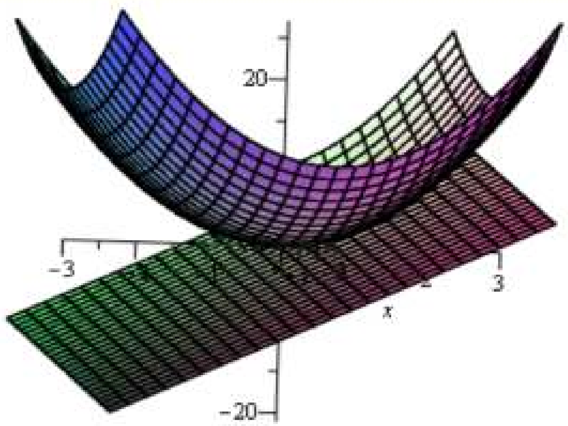
Panel 9

$f(x,y) = 2x^2 + y^2$ Tangent plane at $P(1,1,3)$ is:

$z = 4(x-1) + 2(y-1) + 3$

$plot3d\left(\left\{2\cdot x^2 + y^2, 4\cdot(x-1) + 2\cdot(y-1) + 3\right\}, x=-3..3, y=-3..3\right)$



next Maple

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Panel 10

The Chain Rule

$x = x(t)$: $f(x) = f(x(t)) \Rightarrow \frac{d}{dt} f = \left(\frac{df}{dx}\right)\left(\frac{dx}{dt}\right)$

$f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R} \Rightarrow \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

Chain Rule in $\mathbb{R}^2$:

$z = f(x,y)$, $x = g(t)$, $y = h(t)$

$$\Rightarrow \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Chain Rule in $\mathbb{R}^2$

$z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$

$$\Rightarrow \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

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Panel 11

Ex: $f(x,y) = x^2 y + 3xy^4$, $x = \sin(2t)$, $y = \cos(t)$. Find $\frac{\partial f}{\partial t}$ at $t=0$

without chain rule: $x = \sin(t)$, $y = \cos(t)$, $z = x^2 y + 3xy^4$

$$\Rightarrow z = \sin^2(t)\cos(t) + 3\sin(t)\cos^4(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} = 2\sin(t)\cos^2(t) - \sin^3(t)\cos(t) + 3\cos(t)\cos^4(t) + 3\sin(t)\cdot 4\cos^3(t)\cdot(-\sin(t))$$

$$\left.\frac{\partial z}{\partial t}\right|_{t=0} = 3$$

with chain rule $x = \sin(t)$, $x(0) = 0$; $y(t) = \cos(t)$, $y(0) = 1$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} : \qquad \frac{x=0}{t=0} \qquad \frac{y=1}{t=0}$$

$$\frac{\partial z}{\partial x} = 2xy + 3y^4, \quad \frac{\partial z}{\partial y} = x^2 + 12xy^3, \quad \frac{\partial x}{\partial t} = \cos(t) \to 1, \quad \frac{\partial y}{\partial t} = -\sin(t) \to 0$$

$$\frac{\partial z}{\partial t} = 3\cdot 1 + 0 = 3$$

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Panel 12

Chain Rule is useful for Implicit Differentiation

Suppose $F(x,y) = 0$ is a function defining x and y implicitly, or more precisely, defines $y = y(x)$ implicitly.

$F(x,y)$ and $x = x$, $y = y(x)$

$$\boxed{\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x}}$$

$$0 = \frac{\partial F}{\partial x} 1 + \frac{\partial F}{\partial y} y' \Leftrightarrow 0 = F_x + F_y y'$$

$$\boxed{y' = -F_x / F_y}$$

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Panel 13

Ex: Find y' if $x^3 + y^3 = 6xy$ — $y = y(x)$

① $F(x,y) = x^3 + y^3 - 6xy = 0$ $\left| \frac{\partial}{\partial x} \right.$

② $\frac{\partial}{\partial x}(x^3 + y^3 - 6xy) = \frac{\partial}{\partial x} 0$

$$3x^2 + 3y^2 \cdot \frac{\partial y}{\partial x} - \left(6y + 6x \frac{\partial y}{\partial x}\right) = 0$$

$$3x^2 + 3y^2 y' - 6y - 6x y' = 0$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$\boxed{y' = \frac{6y - 3x^2}{3y^2 - 6x}} = -F_x / F_y$$

$F_x = 3x^2 - 6y$, $F_y = 3y^2 - 6x$

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Panel 14

Technically, not every equation $F(x,y,z) = 0$ defines z as an implicitly diff'ble function: $z = z(x,y)$

Implicit Function Theorem

Suppose F is defined in a sphere around (a,b,c) s.t. $F(a,b,c) = 0$, $F_z(a,b,c) \neq 0$, and F_x, F_y, F_z are continuous inside that sphere. Then

(a) $F(x,y,z) = 0$ defines z as a function of x, y, and

(b) $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are as before!

too complicated → Analysis 1 – Math major

DE for Chem Major Complex Analysis – Math/Physics major

optional

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