Panel 1

Last time

$f: \mathbb{R}^2 \to \mathbb{R}$ e.g. $f(x,y) = x^2 y^2 + \sin(xy)$

$f: \mathbb{R}^3 \to \mathbb{R}$ e.g. $f(x,y,z) = xyz$

Graphs + Slices

Level Curves $f(x,y) = \text{const}$

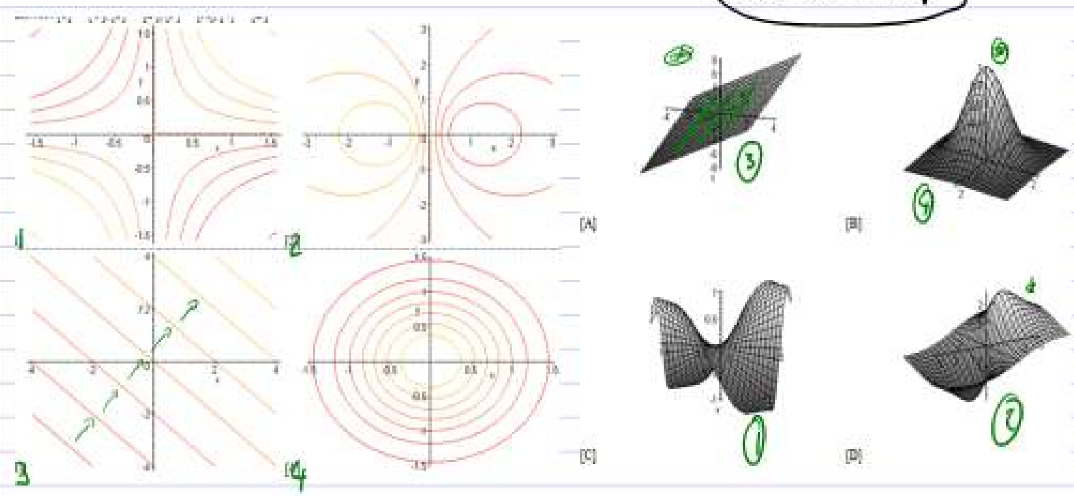
Contour Plots: topographical maps of "same height"

Limits: tricky in $\mathbb{R}^2$

1

Panel 2

Quiz on Friday



Match Contour plots to Graphs

2

Panel 3

Limits $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

Def: Given any $\varepsilon > 0$ there is a $\delta > 0$ such that

Whenever $|(x,y) - (x_0,y_0)| < \delta$ then $|f(x,y) - L| < \varepsilon$

How to really find limits

(1) Plug in and hope for the best

(2) $x = x_0$, $y \to y_0$: L_1

$y = y_0$, $x \to x_0$: L_2

$x = y$, $x \to x_0$: L_3

$x = y^2$ or $y = x^2$: L_4

if any two are different, there is no limit!

(3) ELSE: try to prove that limit is (2)

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Panel 4

Ex: $\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2} = \frac{1}{2}$

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{0}{0} = ?$ d.n.e. ☹

$x = 0$: $\lim_{y\to 0} \frac{0}{y^2} = 0$

$x = y$: $\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

different so

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Panel 5

Ex: Find $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ if it exists no limit

$x=0$: $\lim\limits_{y\to 0} \frac{0}{y^4} = 0$

$y=0$: $\lim\limits_{x\to 0} \frac{0}{x^2} = 0$

$y=\sin(x)$: $\lim\limits_{x\to 0} \frac{x\sin^2(x)}{\dots}$

$y=\cos(x)$: $\lim\limits_{x\to 0} \frac{x\cos^2(x)}{\dots}$

$x=y$: $\lim\limits_{x\to 0} \frac{x^3}{x^2+x^4} = \lim\limits_{x\to 0} \frac{3x^2}{2x+4x^3} = \lim\limits_{x\to 0} \frac{6x}{2+8x^2} = 0$

$y=x^2$: $\lim\limits_{x\to 0} \frac{x^5}{x^2+x^8} = 0$

$x=y^2$: $\lim\limits_{y\to 0} \frac{y^4}{2y^4} = 1/2$

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Panel 6

Ex: $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

$x=0$: $\lim\limits_{y\to 0} \frac{0}{y^2} = 0$

$y=0$: $\lim\limits_{x\to 0} \dots = 0$

$x=y$: $\lim\limits_{x\to 0} \frac{3x^3}{2x^2} = 0$

maybe there is a limit. If so it would have to be 0

$x=y^2$: $\lim\limits_{y\to 0} \frac{3y^5}{y^4+y^2} = 0$

$y=x^2$: $\lim\limits_{x\to 0} \frac{3x^4}{x^2+x^4} = 0$

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Panel 7

Prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

Take $\varepsilon > 0$. Find $\delta > 0$ such that

$|f(x,y) - 0| < \varepsilon$ if $\|(x,y) - (0,0)\| < \delta$

$\left|\frac{3x^2y}{x^2+y^2}\right| < \varepsilon$ if $\sqrt{x^2+y^2} < \delta$

pick $\delta = \frac{1}{3}\varepsilon$ and go backwards

Note: $x^2 \le x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2} \le 1$

$\Rightarrow \frac{3|y|x^2}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2} < ?$

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Panel 8

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ d.n.e.

$x=0, y\to 0$: limit $=0$; $y=0, x\to 0$: limit $=1$

Ex: $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ d.n.e.

$x=0, y\to 0$, limit $=0$; $y=x^2$: limit $=1$

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$ = does exist. HW $\Leftarrow$

Hint: $|x^3| = |x \cdot x^2| = |x|x^2$

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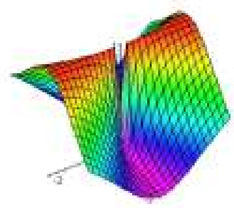
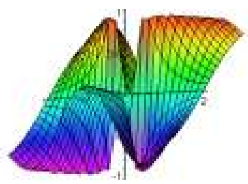
Panel 9

$$f(x,y) = \frac{x^2}{x^2+y^2}$$

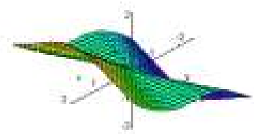


jumps at $(0,0)$

$$f(x,y) = \frac{2x^2y}{x^4+y^2}$$



$$f(x,y) = \frac{x^3}{x^2+y^2}$$



looks smooth at $(0,0)$

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Panel 10

Continuity

As usual, continuity is just a result of limit.

Def: $f(x,y)$ is continuous at (x_0,y_0) if

$$\lim_{(x_0,y_0)\to(x,y)} f(x,y) = f(x_0,y_0)$$

Ex:

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ \cancel{1}\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

not cont. at $(0,0)$

still not cont. at $(0,0)$

Recall: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

NOW it is cont. at $(0,0)$

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Panel 11

Derivatives: 2 vars → 2 derivatives

If $f(x,y)$ is a function of 2 variables, define

fix y: $\lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \frac{\partial f}{\partial x} = f_x$ partial deriv. with respect to x

fix x: $\lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} = \frac{\partial f}{\partial y} = f_y$ partial deriv. with respect to y

∂ = "del" = "De" $\frac{\partial f}{\partial x}$ = "de-f" / "de-x" / $\frac{df}{dx}$

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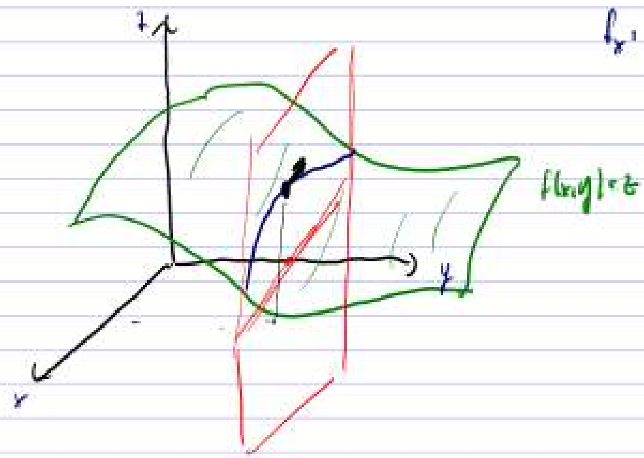
Panel 12

f_x: fix y



f_x is slope of the tangent in direction of x $f(x,y) \cap \{y = \text{const}\}$

f_y is slope of the tangent in direction of y $f(x,y) \cap \{x = \text{const}$

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Panel 13

Ex: Find f_x if $f(x,y) = x^2y + y^2$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{(x+h)^2 y + \cancel{y^2} - (x^2 y + \cancel{y^2})}{h} =$$

How to really do it: pretend other variable is a constant

$$= \lim_{h\to 0} \frac{y\left((x+h)^2 - x^2\right)}{h} =$$

$$= \lim_{h\to 0} \frac{y\left(\cancel{x^2} + 2xh + h^2 - \cancel{x^2}\right)}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{h}\, y(2x+h)}{\cancel{h}} = \lim y(2x+h) =$$

$$= 2xy$$

$$f_y(x,y) = x^2 + 2y$$

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Panel 14

Ex: $f(x,y) = x^3 + x^2y^3 - 2y^2$. Find

$f_x(2,1) = 3\cdot 4 + 2\cdot 2\cdot 1 = 16$

$f_x(x,y) = 3x^2 + 2xy^3$

$f_y(2,1) =$ Mr. Penny man ← HW

$f_y(x,y) = 3x^2y^2 - 4y$

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Panel 15

3D Example: $f(x,y,z) = (xz)(e^{x^2+y^2})$. Find

$f_x(x,y,z) = z\,e^{x^2+y^2} + xz\,2x\,e^{x^2+y^2}$

$f_y(x,y,z) =$ HW

$f_z(x,y,z) = x\,e^{x^2+y^2}$

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Panel 16

Of course we can also take higher-order partial derivatives. Let $f(x,y)$ be a function

1st order: f_x f_y 2

2nd order: f_{xx} f_{xy} f_{yx} f_{yy} 4

3rd deriv: f_{xxx} f_{xxy} f_{xyy} f_{xyy} 8

4th 16

$\vdots$

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Panel 17

Ex: $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x(x,y) = 3x^2 + 2xy^3$

$f_y(x,y) = 3x^2y^2 - 4y$

$f_{xx}(x,y) = 6x + 2y^3$

$f_{xy}(x,y) = 6xy^2$

$f_{yx}(x,y) = 6xy^2$

$\Big\}$ same: $f_{xy} = f_{yx}$

$f_{yy}(x,y) = 6x^2y - 4$

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Panel 18

Notation:

$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f$

$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2} f$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} f$

$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x} f$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2}{\partial y^2} f$

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