Panel 1

Functions of Several Variables

Know. $f: \mathbb{R} \to \mathbb{R}$ e.g. $f(x) = x^2$

(1) $f: \mathbb{R} \to \mathbb{R}^2$ e.g. $r(t) = \langle t^3, t^2 \rangle$

$f: \underline{\mathbb{R}} \to \underline{\mathbb{R}^3}$ e.g. $r(t) = \langle \sin(t), \cos(t), \sin(t) \rangle$

Now: $f: \mathbb{R}^2 \to \mathbb{R}$ e.g. $f(x,y) = xy^2$

(2) $f: \mathbb{R}^3 \to \mathbb{R}$ e.g. $f(x,y,z) = x^2 + y^2 + \sin(z)$

Last $f: \mathbb{R}^2 \to \mathbb{R}^2$ e.g. $f(x,y) = \langle xy, x^2+y^2 \rangle$

$\mathbb{R}^3 \to \mathbb{R}^3$

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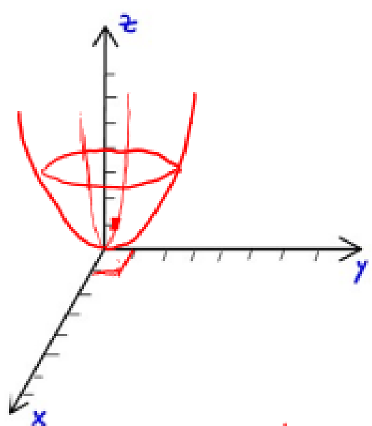
Panel 2

Def: A function of 2 variables is a rule that assigns to every pair (x,y) in a set $D \subset \mathbb{R}^2$ exactly one number $z = f(x,y)$

Ex: $f(x,y) = x^2 + y^2$

x	y	$z = x^2+y^2$
0	0	0
0	1	1
0	-1	1
1	0	1
-1	0	1
1	1	2
⋮	⋮	

$z = x^2 + y^2$



x fixed

$x=0$: parab

$y=0$: parab

$z=1$

complicated.

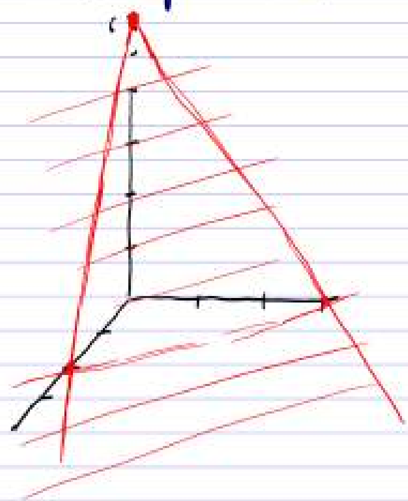
look in "slices"

2

Panel 3

Ex: $f(x,y) = 6 - 3x - 2y$

$z = 6 - 3x - 2y$ is a plane



$y=0$:

$y=0: \quad z = 6 - 3x$

$x=0: \quad z = 6 - 2y$

$z=0: \quad 0 = -3x - 2y$

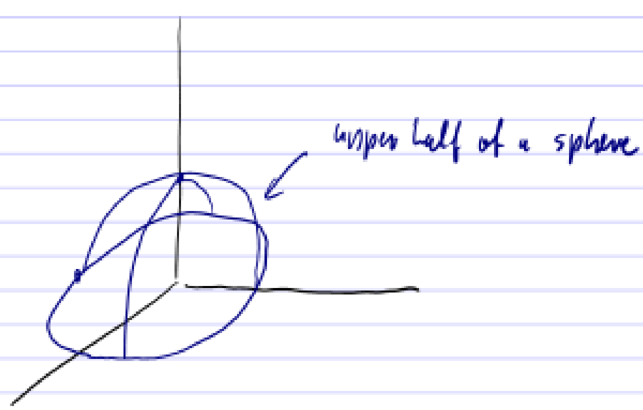
3

Panel 4

Ex: $f(x,y) = \sqrt{9 - x^2 - y^2}$

$z = \sqrt{9 - x^2 - y^2} \Leftrightarrow z^2 = 9 - x^2 - y^2$

$\Leftrightarrow x^2 + y^2 + z^2 = 9 \Rightarrow$



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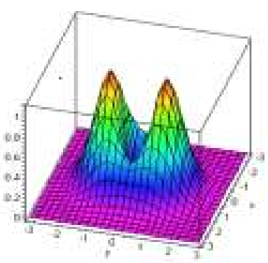
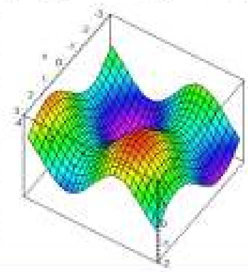
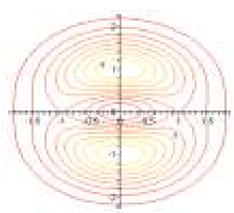
Panel 5

$f(x,y) = (x^2+3y^2)\,e^{-x^2-y^2}$ $f(x,y) = \sin(x) + \sin(y)$

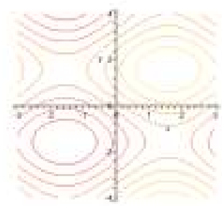




<u>level curves:</u> curves of constant height, i.e. $f(x,y) = z = \text{constant}$



contour plots or topographical maps



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Panel 6

Of course I used Maple to generate these plots

```
> plot3d((x^2+3*y^2)*exp(-x^2-y^2),x=-3..3,y=-4..4);
> plot3d(sin(x)+sin(y),x=-3..3,y=-4..4);
> with(plots);
> contourplot((x^2+3*y^2)*exp(-x^2-y^2),x=-3..3,y=-4..4);
> contourplot(sin(x)+sin(y),x=-3..3,y=-4..4);
>
```

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Panel 7

Ex: Level curves of $h(x,y) = 4x^2+y^2 = 1$ or 2 or 3 or 4

ellipses

Ex: Level curves for $f(x,y) = 6-3x-2y = 1$

lines

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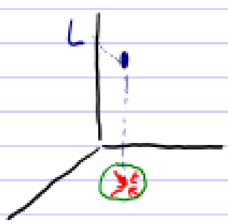
Panel 8

Limits: The limit of $f(x,y)$ as (x,y) approaches (x_0,y_0) is L is written as



$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad , f:\mathbb{R}^2 \to \mathbb{R}$$

Given $\varepsilon > 0$ there is a $\delta > 0$ s.t.

Limits are tricky, because I can get close to (x_0,y_0) from any direction

if $\|(x,y)-(x_0,y_0)\| < \delta$ then $|f(x,y)-L| < \varepsilon$

if $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$ then $|f(x,y)-L| < \varepsilon$

$\sqrt{(x-x_0)^2+(y-y_0)^2} = \delta$ is a

is a circle, centered at (x_0,y_0), radius δ

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Panel 9

In $\mathbb{R}$: $\lim_{x \to x_0} f(x)$

left / right handed limit

In $\mathbb{R}^2$: $\lim_{(x,y) \to (x_0,y_0)} f(x,y)$

inf. many ways to get close to (x_0,y_0)

PROBLEM!

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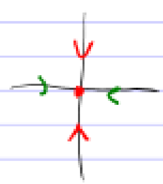
Panel 10

Hints for finding limits in $\mathbb{R}^2$:

→ if C_1 is a path to (x_0,y_0) and $f(x,y) \to L_1$ on C_1

→ if C_2 is a path to (x_0,y_0) and $f(x,y) \to L_2$ on C_2

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist



$x=0, y \to 0$: $\lim_{y \to 0} \frac{-\cancel{y^2}}{\cancel{y^2}} = -1$ } different, so

$y=0, x \to 0$: $\lim_{x \to 0} \frac{x^2}{x^2} = 1$ no limit!

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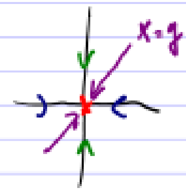
Panel 11

Ex: $\lim\limits_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2} = \frac{1}{2}$

$\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$



Let: $x=0,\ y\to 0$: $\lim\limits_{y\to 0} \frac{0}{y^2} = 0$

$y=0,\ x\to 0$: $\lim\limits_{x\to 0} \frac{0}{x^2} = 0$

$x=y\to 0$: $\lim\limits_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

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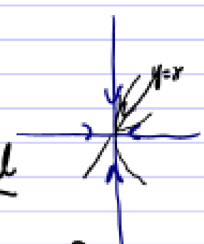
Panel 12

Ex: Find $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ if it exists

$x=0,\ y\to 0$: $\lim\limits_{y\to 0} \frac{0}{y^4} = 0$

$y=0,\ x\to 0$: $\lim\limits_{x\to 0} \frac{0}{x^2} = 0$



L'Hospital

$y=x$: $\lim\limits_{x\to 0} \frac{x^3}{x^2+x^4} = \lim \frac{3x^2}{2x+4x^3} = \lim \frac{6x}{2+12x^2} = 0$

$x=y^2$: $\lim\limits_{x\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2}$ no limit

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Panel 13

Ex: $\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2+y^2}$ try this at home

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