Panel 1

Last Time we discussed vector-valued functions

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. We talked about:

Tangent vectors: $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$ ✓

Unit tangent vectors: $\vec{T}(t) = \vec{r}\,'(t) / \|\vec{r}\,'(t)\|$ ✓

Length: $s = \int_a^b \sqrt{(f')^2 + (g')^2 + (h')^2}\, dt = \int_a^b \|\vec{r}\,'(t)\|\, dt$

Curvature: $\kappa = \dfrac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}$

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Panel 2

Ex. Find the curvature for $\vec{r}(t) = \langle t, t^2 \rangle$

$\kappa = \dfrac{\|T'(t)\|}{\|r'(t)\|}$ so we need r' and T first:

$r(t) = \langle t, t^2 \rangle \Rightarrow r'(t) = \langle 1, 2t \rangle \Rightarrow \|r'(t)\| = \sqrt{1+4t^2}$

Thus $T(t) = r'(t) / \|r'(t)\| = \dfrac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle$

Note that T almost always involves a square root.

Now: $T'(t) = \dfrac{d}{dt}\left((1+4t^2)^{-1/2} \cdot \langle 1, 2t \rangle \right) =$ product rule

$-\frac{1}{2}(1+4t^2)^{-3/2} \cdot 8t \langle 1, 2t \rangle + (1+4t^2)^{-1/2} \langle 0, 2 \rangle =$ trouble

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Panel 3

$r(t) = \langle t, t^2 \rangle \Rightarrow$ curvature $\kappa = \dfrac{2}{(1+4t^2)^{3/2}}$ $\cdot \dfrac{\|T'\|}{\|r'\|}$

Finding the curvature requires finding the derivative T' of the unit tangent T. Since T usually involves square roots, it is almost always painful to find T' and therefore the curvature. There is, however, a short cut:

Theorem: If $r: \mathbb{R}^3 \to \mathbb{R}$ is a 3D vector-valued function, the curvature κ can be found:

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3}$$ much easier

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Panel 4

In our previous example $r(t) = \langle t, t^2 \rangle$

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} \qquad r(t) = \langle t, t^2, 0 \rangle$$

$r' = \langle 1, 2t, 0 \rangle.$

$r'' = \langle 0, 2, 0 \rangle$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

$$\Rightarrow \kappa = \frac{\|\langle 0,0,2 \rangle\|}{\|\langle 1, 2t, 0 \rangle\|^3} = \frac{2}{(\sqrt{1+4t^2})^3} = \frac{2}{(1+4t^2)^{3/2}}$$

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Panel 5

To continue we first need a theoretical result:

Thm: If $r(t)$ is a vector-valued function such that $\|r(t)\| = 1$ $\forall t$ then $\underline{r(t)}$ is perp. to $\underline{r'(t)}$.

Proof: Need: $r' \cdot r = 0$. So

$$1 = \|r\|^2 = r \cdot r \quad | \, d/dt$$

$$0 = \frac{d}{dt} r \cdot r = r' r + r r' = 2 r r' =$$

$$\Rightarrow \underline{r' \cdot r = 0}$$

#

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Panel 6

Since for the unit tangent $T(t)$ we have $\|T\| = 1$ we know that T and T' are perpendicular to each other.

Ex: For $\vec{r}(t) = \langle t, t^2 \rangle$ we had:

$$T(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle, \quad T'(t) = \frac{1}{(1+4t^2)^{3/2}} \langle -4t, 2 \rangle$$

Thus $T \cdot T' = \frac{1}{(1+4t^2)^{1/2}} \langle 1, 2t \rangle \cdot \frac{1}{(1+4t^2)^{3/2}} \langle -4t, 2 \rangle = \frac{1}{(1+4t^2)^2} \cdot \vec{0} = \vec{0}$

We normalize the vector T' and give it a new name: principle normal

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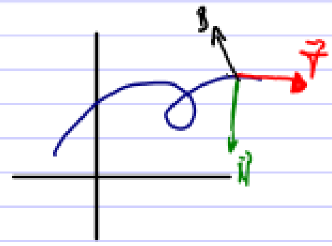
Panel 7

Def: If $T(t)$ is the unit tangent to a space curve $r(t)$ then $T'(t)$ is perpendicular to $T(t)$. We define:



$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$ principle normal

$$B(t) = T \times N$$ binormal

T, N, B form a local coord. system to $\vec{r}(t)$ $\forall t$ ← "for all"

Panel 8

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find tangent, unit normal and binormal vectors.

$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

$\Rightarrow \vec{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$

$\Rightarrow T'(t) = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle$, $\|T'\| = 1/\sqrt{2}$

$\Rightarrow \vec{N}(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

$$\Rightarrow \vec{B}(t) = \begin{vmatrix} i & j & k \\ -\frac{1}{\sqrt{2}}\sin(t) & \frac{1}{\sqrt{2}}\cos(t) & 1/\sqrt{2} \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} = \langle +\tfrac{1}{\sqrt{2}}\sin(t), -\tfrac{1}{\sqrt{2}}\cos(t), \tfrac{1}{\sqrt{2}} \rangle$$

$$= \frac{1}{\sqrt{2}} \langle +\sin(t), -\cos(t), 1 \rangle$$

Panel 9

$r(0) = \langle 1, 0, 0 \rangle$

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find unit tangent, unit normal and binormal vectors at $t = 0$

$T(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$

$N(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

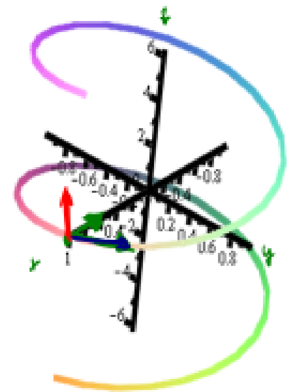
$B(t) = \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle$

$t = 0$:

$\Rightarrow T = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle$ ✓

$\Rightarrow N = \langle -1, 0, 0 \rangle$

$\Rightarrow B = \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle$



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Panel 10

Summary

$r(t) = \langle f(t), g(t), h(t) \rangle$ space curve

$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$ tangent vector

$T(t) = r'(t) / \|r'(t)\|$ unit tangent

$N(t) = T' / \|T'\|$ principle normal

$B(t) = T \times N$ binormal

$\kappa = \frac{\|T'\|}{\|r'\|} = \frac{\|r' \times r''\|}{\|r'\|^3}$ curvature

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Panel 11

Quiz 4

Suppose $\vec{r}(t) = \langle t^2, 2, t\rangle$ is a vector-valued function (aka space curve), representing the position of a particle. Find the following:

1. The velocity r' at $P(0,2,0)$ $\Leftrightarrow$ $t=0$
2. The speed at $P(0,2,0)$ $\|r'\|$
3. The acceleration r'' at $P(0,2,0)$
4. The unit tangent $\vec{T}(t)$ at $P(0,2,0)$ $t=0$
5. The unit normal vector $\vec{N}(t)$ at $P(0,2,0)$ terrible
6. The bi-normal vector $\vec{B}(t)$ at $P(0,2,0)$
7. The curvature k at $P(0,2,0)$

$r(t) = \langle 4t+1, t, 7t\rangle$

$P(5,1,7) \Leftrightarrow t=1$

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Panel 12

Exam on Monday

$$r(t) = \langle t^2, 2, t\rangle$$

$$r'(t) = \langle 2t, 0, 1\rangle$$

$$\|r'(t)\| = \sqrt{1+4t^2}$$

$$r''(t) = \langle 2, 0, 0\rangle$$

$$T(t) = (1+4t^2)^{-1/2}\cdot\langle 2t, 0, 1\rangle, \quad T(0) = \langle 0, 0, 1\rangle$$

$$T'(t) = -\tfrac{1}{2}(1+4t^2)^{-3/2}\cdot 8t\,\langle 2t, 0, 1\rangle + (1+4t^2)^{-1/2}\langle 2, 0, 0\rangle =$$

$$= \left\langle \frac{-8t^2}{(1+4t^2)^{3/2}} + \frac{2}{(1+4t^2)^{1/2}}, 0, \frac{-4t}{(1+4t^2)^{3/2}}\right\rangle =$$

$$= \left\langle \frac{-8t^2 + 2(1+4t^2)}{(1+4t^2)^{3/2}}, 0, -4t/(1+4t^2)^{3/2}\right\rangle = \frac{1}{(1+4t^2)^{3/2}}\langle 2, 0, -4t\rangle$$

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Panel 13

$$T' = \frac{2}{(1+4t^2)^{3/2}} \langle 1, 0, -2t \rangle$$

$$\|T'\| = \frac{2}{(1+4t^2)^{3/2}} \sqrt{1+4t^2} = \frac{2}{(1+4t^2)}$$

$$N = \frac{2}{(1+4t^2)^{3/2}} \langle 1, 0, -2t \rangle \cdot \frac{(1+4t^2)}{2} = \frac{1}{(1+4t^2)^{1/2}} \langle 1, 0, -2t \rangle$$

$$N(0) = \langle 1, 0, 0 \rangle$$

$$B = T \times N = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, +1, 0 \rangle$$

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Panel 14

Def: The plane determined by T and N is called osculating plane or supporting plane.

extra credit

(Latin: osculum = kiss)

Def: The circle in the osculating plane with radius $r = 1/\kappa$ is called the osculating circle.

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Panel 15

Motion in Space

Suppose $\vec{r}(t)$ represents the motion of an object through space over time

$\vec{r}(t)$ = path or motion

$r'(t)$ = velocity , $r'(t) = \vec{v}(t)$

$\|r'(t)\|$ = speed , $\|r'(t)\| = s(t)$

$r''(t)$ = acceleration , $r''(t) = \vec{a}(t)$

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Panel 16

Ex: Suppose the path of a particle at time t is $\vec{r}(t) = \langle t^3, t^2 \rangle$. Find velocity, speed, and acceleration at $P(1,1)$. Illustrate.

$v(t) = \langle 3t^2, 2t \rangle \Rightarrow v(1) = \langle 3, 2 \rangle$

$s(t) = \sqrt{9t^4 + 4t^2} \Rightarrow s(1) = \sqrt{13}$

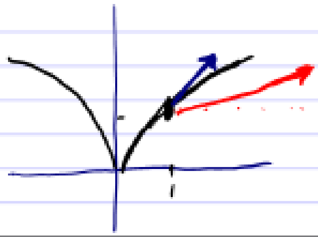
$\vec{a}(t) = \langle 6t, 2 \rangle \Rightarrow a(1) = \langle 6, 2 \rangle$

$r(t) = \langle t^3, t^2 \rangle$

$x = t^3, y = t^2$

$x^{1/3} = t, y = (x^{1/3})^2$

$y = x^{2/3}$



$\vec{a}$ needs to push forward and "down to right"

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Panel 17

Ex: A particle starts at $P(1,0,0)$ with initial velocity $\langle 1,-1,1\rangle$. The acceleration is $\vec{a}(t)=\langle 4t,6t,1\rangle$. Find velocity, speed, and position.

$$\vec{a}(t)=\langle 4t,\ 6t,\ 1\rangle$$

$$\vec{v}(t)=\int \vec{a}(t)\,dt=\langle 2t^2,\ 3t^2,\ t\rangle+\langle c_1,c_2,c_3\rangle$$

$$\text{Know } \vec{v}(0)=\langle 1,-1,1\rangle \Rightarrow \vec{v}(t)=\underline{\langle 2t^2,3t^2,t\rangle}+\langle 1,-1,1\rangle$$

$$\vec{r}(t)=\int \vec{v}(t)\,dt=\langle \tfrac{2}{3}t^3,\ t^3,\ \tfrac{1}{2}t^2\rangle+\langle t,-t,t\rangle+\underline{\langle 1,0,0\rangle}$$

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Panel 18

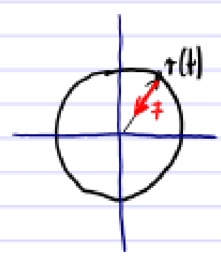
Ex: An object with mass m moves in a circle with constant angular speed ω. Find the force acting on the object and illustrate.

$$r(t)=\langle \cos(\omega t),\ \sin(\omega t)\rangle$$

$$r'(t)=\langle -\omega\sin(\omega t),\ \omega\cos(\omega t)\rangle,\quad s=\|r'(t)\|=\omega$$

$$\|r'(t)\|=\sqrt{\omega^2\sin^2(\omega t)+\omega^2\cos^2(\omega t)}=\sqrt{\omega^2\cdot 1}=\omega$$

$$r''(t)=\langle -\omega^2\cos(\omega t),\ -\omega^2\sin(t)\rangle$$



$$\vec{F}=m\vec{a}=-m\omega^2\langle \cos(\omega t),\sin(\omega t)\rangle$$

$$=-m\omega^2\,\vec{r}(t)$$

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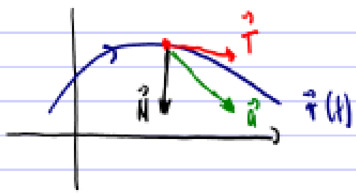
Panel 19

Tangential and Normal Components of Acceleration



acceleration has two parts:

- in direction of T to speed up or slow down particle
- in direction of N to change direction

$$\vec{a} = a_t \vec{T} + a_N \vec{N}$$

a_t = tangential comp. of accel.

a_N = normal comp. of acceleration

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Panel 20

Theorem: $\vec{a} = a_T \vec{T} + a_N \vec{N}$ where

tang. component $a_t = \dfrac{\vec{v} \cdot \vec{a}}{s}$

normal component $a_N = \dfrac{\| \vec{v} \times \vec{a} \|}{s}$

Ex: $r(t) = \langle t^2, t^2, t^3 \rangle$ - find a_t and a_N

on HW

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Panel 21

Quiz 4

Suppose $\vec{r}(t) = \langle t^2, 2, t \rangle$ is a vector-valued function (aka space curve), representing the position of a particle. Find the following:

1. The velocity at $P(0,0,0)$
2. The speed at $P(0,0,0)$
3. The acceleration at $P(0,0,0)$
4. The unit tangent $\vec{T}(t)$ at $P(0,0,0)$
5. The unit normal vector $\vec{N}(t)$ at $P(0,0,0)$
6. The bi-normal vector $\vec{B}(t)$ at $P(0,0,0)$
7. The curvature k at $P(0,0,0)$
8. The tangential component of the acceleration a_T at $P(0,0,0)$
9. The normal component of the acceleration a_N at $P(0,0,0)$
10. The osculating plane at $P(0,0,0)$
11. The osculating circle at $P(0,0,0)$

Extra 10, 11

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