Panel 1

Last time

Vector-valued functions: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$r: \mathbb{R} \to \mathbb{R}^3$ or $r: \mathbb{R} \to \mathbb{R}^2$

Derivatives: $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Dot-Product Rule $\frac{d}{dt} \vec{v}(t) \cdot \vec{w}(t) = \vec{v}' \cdot \vec{w} + \vec{v} \cdot \vec{w}'$

Note: $\vec{v} \cdot \vec{w} : \mathbb{R} \to \mathbb{R}$

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Panel 2

Integrals: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ then

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$$

Ex: $\vec{r}(t) = 2\cos(t)\vec{i} + \sin(t)\vec{j} + 2t\vec{k}$ find

$$\int_0^{\pi/2} \vec{r}(t)\,dt = \left\langle \int_0^{\pi/2} 2\cos(t)\,dt, \int_0^{\pi/2} \sin(t)\,dt, \int_0^{\pi/2} 2t\,dt \right\rangle =$$

$$= \left\langle 2\sin(t)\Big|_0^{\pi/2}, -\cos(t)\Big|_0^{\pi/2}, t^2\Big|_0^{\pi/2} \right\rangle =$$

$$= \left\langle 2, -(0-1), \frac{\pi^2}{4} \right\rangle = \left\langle 2, +1, \frac{\pi^2}{4} \right\rangle$$

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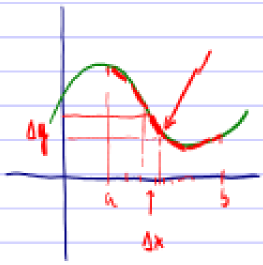
Panel 3

Arc Length:



$$\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta t^2\left[\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2\right]}$$

$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$$

$\frac{x(t_1)-x(t_2)}{t_1-t_2} \to x'(t_1)$, $\lim_{t_1\to t_2}$

$$\text{total length} \approx \sum_{j=1}^{n} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$$

$$\underline{\text{length}} = \lim_{n\to\infty} \text{ - above - } = \int_a^b \sqrt{(x')^2+(y')^2}\,dt$$

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Panel 4

Theorem: The length of a continuous vector valued function $r(t) = \langle f(t), g(t), h(t)\rangle$ as t goes from a to b is

value $\to$ $s = \int_a^b \sqrt{(f')^2+(g')^2+(h')^2}\,dt$ ($\leftarrow$ calc 2 $\int \sqrt{1+(f')^2}\,dx$)

Ex: Length of $\langle r\cos(t), r\sin(t)\rangle$, $t\in[0,2\pi]$

$$s = \int_0^{2\pi}\sqrt{r^2\sin^2(t)+r^2\cos^2(t)}\,dt = 1\cdot 2\pi r = \underline{2\pi}\, r$$

Makes sense, because $r(t)=\langle\cos(t),\sin(t)\rangle$ is unit circle!

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Panel 5

Length of a slinky around z-axis? Once around

$$r(t) = \langle \cos(t), \sin(t), t \rangle$$

$$\Rightarrow s = \int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t) + 1}\, dt = \underline{\underline{\sqrt{2}\, 2\pi}}$$

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Panel 6

t in r(t) stands for time: many possible parametrizations of a curve:

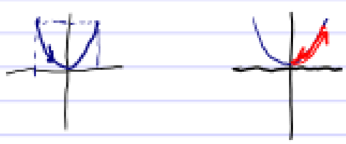
$$\vec{r}_1(t) = \langle \overset{x}{t}, \overset{y}{t^2} \rangle \quad y = x^2 \quad , t = 0..1$$

$$\vec{r}_2(t) = \langle \overset{x}{t^3}, \overset{y}{t^6} \rangle \quad y = x^2 \quad , t = 0..1$$

are the same curves but traversed at different speeds

<u>Careful:</u> $\langle t, t^2 \rangle$ $\langle t^2, t^4 \rangle$ on $t = 0..1$ are the same

but $\langle t, t^2 \rangle$, $\langle t^2, t^4 \rangle$, on $t = -1..1$ are different



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Panel 7

Let $r_1(t) = \langle t, t^2, t^3 \rangle$, as $t = 1$ to 2

$r_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle$, as $u = 0$ to $u = \ln(2)$ describe same curve

$e^u = 2 \Rightarrow u = \ln(2)$

Are their lengths the same?

$r_1: \quad s = \int_1^2 \sqrt{1 + (2t)^2 + (3t^2)^2}\, dt$ $\qquad r_2: \quad s = \int_0^{\ln(2)} \sqrt{(e^u)^2 + (2e^{2u})^2 + (3e^{3u})^2}\, du$

Subst: $t = e^u \Rightarrow dt = e^u\, du$

$$\int_0^{\ln(2)} \sqrt{1 + (2e^u)^2 + (3e^{2u})^2}\; e^u\, du$$

//

$$\int_0^{\ln(2)} \sqrt{e^{2u}(1 + 4e^{2u} + 9u^{4u})}\, du = \int_0^{\ln(2)} \sqrt{e^{2u} + 4e^{4u} + 9e^{6u}}\, du$$

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Panel 8

Ex: Compute length of $r(t) = \langle t, \sqrt{1-t^2} \rangle$, t from -1 to 1

$x'(t) = 1$, $\quad y'(t) = \frac{1}{2}(1-t^2)^{-1/2} \cdot (-2t) = -\frac{t}{\sqrt{1-t^2}}$

$\langle t, \sqrt{1-t^2} \rangle$

$x = t$,

$y = \sqrt{1-t^2} = \sqrt{1-x^2}$

$y^2 = 1 - x^2$, $x^2 + y^2 = 1$

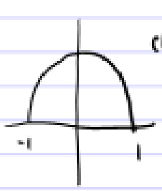
$$s = \int_{-1}^{1} \sqrt{1 + \frac{t^2}{1-t^2}}\, dt = \int_{-1}^{1} \sqrt{\frac{1-t^2+t^2}{1-t^2}}\, dt = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, dt$$

Let $t = \sin(u)$

$dt = \cos(u)\, du$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{1 - \sin^2(u)}} \cos(u)\, du$$

$$= \int_{-\pi/2}^{\pi/2} 1\, du = u \Big|_{-\pi/2}^{\pi/2} = \pi$$



choose $\langle \cos(t), \sin(t) \rangle$

$t = 0, \pi$

$\frac{d}{dx} \arcsin(x) = 1/\sqrt{1-x^2}$

$\frac{d}{dx} \arctan(x) = 1/(1+x^2)$

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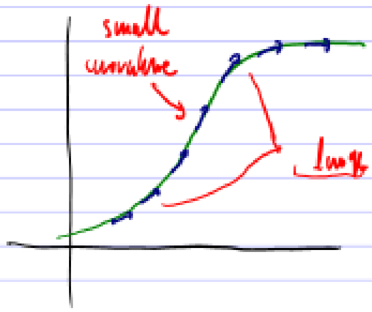
Panel 9

We can measure direction of a curve (derivative) and length of a curve. Want to measure curvature:



large curvature should be define as

"change in tangent".

Curvature $\kappa = \left\| \frac{dT}{dt} \right\|$ where T is unit tangent vector

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Panel 10

Def. If $\vec{r}(t)$ is a vector valued function, then:

$$T(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

is the unit tangent vector to the curve

Def: The curvature of a vector valued function is

$$\kappa = \left\| \frac{dT}{dt} \right\| = \frac{\|T'(t)\|}{\|\vec{r}\,'(t)\|}$$

theorem

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Panel 11

Ex: $r(t) = \langle t, t^2 \rangle$. Find $T(1)$

$r'(t) = \langle 1, 2t \rangle$

$\|r'(t)\| = \sqrt{1+4t^2}$

$\Rightarrow T(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle \Rightarrow T(1) =$

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Panel 12

Ex: What is the curvature of a circle of radius r?

$r(t) = \langle r\cos(t), r\sin(t) \rangle$, $t \in (0, 2\pi)$

$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|}$ $\qquad r'(t) = \langle -r\sin(t), r\cos(t) \rangle$

$\Rightarrow T(t) = \frac{r'(t)}{\|r'(t)\|} \langle -\sin(t), \cos(t) \rangle$

$T'(t) = \langle -\cos(t), -\sin(t) \rangle \Rightarrow \|T'(t)\| = 1$

$\Rightarrow \kappa = \frac{1}{r} \cdot 1 = 1/r$ constant $\to$ no surprise

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