Panel 1

Last time

Dot Product

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Geometrically

$\vec{a}\cdot\vec{b} = \|a\|\|b\|\cos(\theta)$

$a\cdot b = 0 \iff a \perp b$

Projection:

$proj_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{\|a\|^2}\vec{a}$ ✓

Cross Product

$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle\ \ , -(\ \ ), \ \ \rangle$

Geometrically

$\|\vec{a}\times\vec{b}\| = \|a\|\|b\|\sin\theta$

$a\times b$ is perp. to both $\vec{a}, \vec{b}$

Volume of Parallelogram

X

1

Panel 2

Last Line Really:

$\underline{l}(t) = P_0 + t\vec{v}$, $\vec{v}$ direction of the line

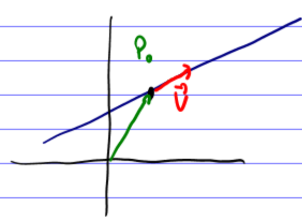
P_0 is a point on the line

$\iff \langle x,y,z\rangle = (x_0,y_0,z_0)\ t + \langle v_1,v_2,v_3\rangle$

$x = x_0 + tv_1$

$y = y_0 + tv_2$

$z = z_0 + tv_3$



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Panel 3

Ex: Suppose 2 lines are $\ell_1(t) = \langle 1+t, -2+3t, 4-t \rangle$

$\ell_2(s) = \langle 2s, 3+s, -3+4s \rangle$

a) Are the lines parallel?

direction of ℓ_1: $\vec{v} = \langle 1, 3, -1 \rangle$

ℓ_2: $\vec{v} = \langle 2, 1, 4 \rangle$ $\rightarrow$ not parallel!

b) Do they intersect in $\mathbb{R}^3$? If so: $\ell_1(t) = \ell_2(s)$

$1+t = 2s \quad \rightarrow t = 2s-1 = 2 \cdot \frac{8}{5} - 1 = \frac{16}{5} - 1 = \frac{11}{5}$

$-2+3t = 3+s \quad -2 + 3(2s-1) = 3+s \Leftrightarrow -2+6s-3 = 3+s$

$4-t = -3+4s \qquad -8 = -5s \rightarrow s = \frac{8}{5}$

$4 - \frac{11}{5} = -3 + 4 \cdot \frac{8}{5}$

$\frac{9}{5} = \frac{17}{5}$ False

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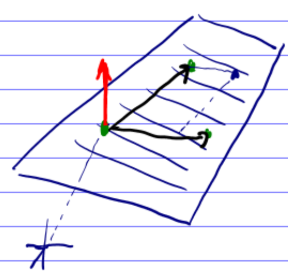
Panel 4

Planes in $\mathbb{R}^3$



A plane in $\mathbb{R}^3$ is uniquely determined by

a) 3 points or

b) 2 vectors and one point

c) one vector perp. to plane + one point

$\ell(t) = P_0 + t\vec{v}$

$p(s,t) = P_0 + t\vec{v} + s\vec{w}$ vector equation of plane, or

parametric equation $\rightarrow$ not that useful

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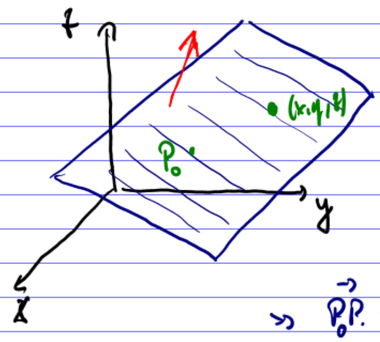
Panel 5

Suppose a plane goes through $P_0(x_0, y_0, z_0)$ and has normal vector $\langle a, b, c \rangle$, i.e. $\langle a, b, c \rangle$ is perp. to the plane



take $P(x,y,z) \in$ plane

$P_0P = \langle x-x_0, y-y_0, z-z_0 \rangle$ is a vector in plane

$\Rightarrow \vec{P_0P} \cdot \vec{n} = 0 \iff \langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$ax + by + cz \underbrace{- ax_0 - by_0 - cz_0}_{d} = 0$

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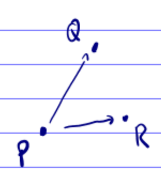
Panel 6

Def: The equation of a plane with normal vector $\vec{n} = \langle a, b, c \rangle$ through the point $P_0(x_0, y_0, z_0)$ is:

① $ax + by + cz + d = 0$ ② or $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Plane through $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$. Find its equation.



$\vec{PQ} = \langle 2, -4, 4 \rangle \quad \vec{PR} = \langle 4, -1, -2 \rangle$

$\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 8+4, -(-4-16), -2+16 \rangle$

$= \langle 12, 20, 14 \rangle \sim 2\langle 6, 10, 7 \rangle$

$\langle a, b, c \rangle = \langle 6, 10, 7 \rangle$

$\nearrow 6(x-1) + 10(y-3) + 7(z-2) = 0$

$\searrow 6x + 10y + 7z + d = 0 \quad \Rightarrow d = \underline{\qquad}$

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Panel 7

Scalar equation of Plane through $P(x_0,y_0,z_0)$ with normal vector $\vec{n} = \langle a,b,c\rangle$ is $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

Ex: Angle between planes (1) $x+y+z=1$ and (2) $x-2y+3z=1$

$n_1 = \langle 1,1,1\rangle$, $n_2 = \langle 1,-2,3\rangle$ are not parallel

angle between planes is the angle between normal vectors

$$\Rightarrow \cos(\theta) = \frac{n_1\cdot n_2}{\|n_1\|\|n_2\|} = \frac{\langle 1,1,1\rangle\cdot\langle 1,-2,3\rangle}{\sqrt{3}\sqrt{14}} = \frac{1-2+3}{\sqrt{3}\sqrt{14}} = 2/\sqrt{3\cdot 14}$$

$$\theta = \cos^{-1}\left(2/\sqrt{3\cdot 14}\right)$$

They intersect in a line. Find it! HW

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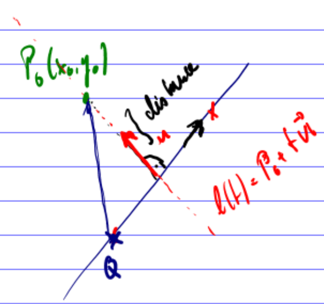
Panel 8

Ex: What is the distance between a point $P_0(x_0,y_0)$ and the line given by $\boxed{ax+by+c=0}$

old fashioned line



Need: 2 points on line:

$x=0 \Rightarrow y=-c/b$, $y=0, x \Rightarrow -c/a$

$(0,-c/b)$ $(-c/a,0)$ are on line

$\rightarrow$ direction $\vec{v} = \langle -c/a, c/b\rangle \cdot \frac{a\cdot b}{c} = \langle -b, a\rangle$

The normal vector (perp. to $\langle -b,a\rangle$) is: $\vec{n} = \langle a,b\rangle$ is normal to the line

distance. take vector from $(0,-c/b)$ to (x_0,y_0) $\Rightarrow \langle x_0, y_0 + c/b\rangle = \vec{PQ}$

$$\left|\text{proj}_{\vec{n}}(\vec{PQ})\right| = \left\|\frac{\vec{n}\cdot\vec{PQ}}{\|\vec{n}\|^2}\vec{n}\right\| = \left|\frac{\langle a,b\rangle\cdot\langle x_0, y_0+c/b\rangle}{\|n\|}\right| \left\|\frac{\vec{n}}{\|n\|}\right\| = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$

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Panel 9

Distance of $P_0(x_0,y_0)$ to line $ax+by+c=0$ is

$$d = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$

Ex: HW

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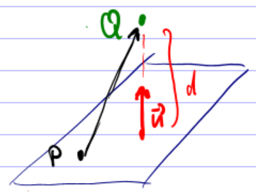
Panel 10

Distance between Point $Q(x_1,y_1,z_1)$ and Plane



$ax+by+cz+d=0$

① Find any point $P(x_0,y_0,z_0)$ on plane

② Find $\vec{PQ}$

③ $\|\text{proj}_{\vec{n}} \vec{PQ}\|$ is answer.

Ex: $10x+2y-2z=5$ distance to origin. $Q=(0,0,0)$

① $(\frac{1}{2},0,0)=P$ ② $\vec{PQ} = \langle \frac{1}{2},0,0\rangle$

③ $\|\text{proj}_{\vec{n}} \vec{PQ}\| = \frac{|\vec{n}\cdot\vec{PQ}|}{\|n\|} = \frac{\langle 10,2,-2\rangle\cdot\langle \frac{1}{2},0,0\rangle}{\sqrt{100+4+4}} = \frac{5}{\sqrt{108}}$

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Panel 11

Find distance between

a) $10x + 2y - 2z = 5$ and $x + y + z = 1$ $\Rightarrow$ $\boxed{\text{dist} = 0}$

b) $10x + 2y - 2z = 5$ and $5x + y - z = 1$ parallel

(1) Any point on plane 1

Any point on plane 2

(2) Proj of that vector onto $\vec{n}$

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Panel 12

Proof Distance between $Q(x_1, y_1, z_1)$ and $ax + by + cz + d = 0$

is $$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

HW

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