Panel 1

Last time

Dot Product

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \in \mathbb{R}$

Geometrically

$\vec{a} \cdot \vec{b} = \|a\| \|b\| \cos(\theta)$

$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

Projection:

$\text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\|a\|^2} \vec{a}$

Cross Product

$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$

$\langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1 \rangle \in \mathbb{R}^3$

Geometrically

$\|\vec{a} \times \vec{b}\| = \|a\| \|b\| \sin(\theta)$

$\vec{a} \times \vec{b}$ perp. to both a and b

$a \cdot (a \times b) =$

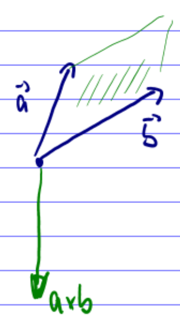
area ~~Volume~~ of Parallelogram

$\|a \times b\|$

1

Panel 2

Picture Problems



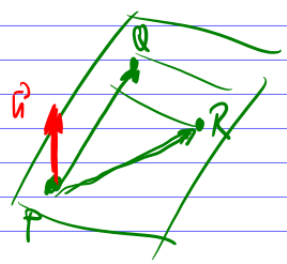
$\vec{a} \cdot \vec{b}$ { positive / zero / negative

$\vec{a} \times \vec{b}$

2

Panel 3

Ex: Find vector perpendicular to the plane through $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$



$\vec{n} = \vec{PQ} \times \vec{PR} =$

$\langle -3, 1, -7\rangle \times \langle 0, -5, -5\rangle$

$$\begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \langle -5-35, -(15\ \ ), 15\rangle =$$

$$= \boxed{\langle -40, -15, 15\rangle}$$

$$5\boxed{\langle -8, -3, 3\rangle}$$

angle between $\vec{PQ}$ and $\vec{PR}$

$\boxed{\langle -3, 1, -7\rangle} \cdot \langle 0, -5, -5\rangle =$

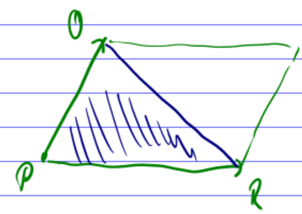
$0 \quad -5 + 35 = \underline{30}$

$\cos(\theta) = 30 / \sqrt{59}\sqrt{50}$

3

Panel 4

Ex: Area of triangle $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$



Area of triangle is ½ area of parallelogram

Parall: $\|\vec{PQ} \times \vec{PR}\| = \langle -40, -15, 15\rangle =$

$= \sqrt{40^2 + 15^2 + 15^2}$

Answer: $\frac{1}{2}\sqrt{40^2 + 15^2 + 15^2}$

4

Panel 5

Name:

Quiz #2

① Suppose $\vec{v} = \langle -1, -2, 3 \rangle$ and $\vec{w} = \langle 1, -4, -2 \rangle$

a) Find $\vec{v} \cdot \vec{w}$

b) Find the angle θ (or $\cos(\theta)$) between $\vec{v}$ and $\vec{w}$

c) Find $\text{proj}_{\vec{v}}(\vec{w})$

5

Panel 6

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta$

$\vec{a} = \langle -3, 1, -7 \rangle$

$\vec{b} = \langle 0, -5, -5 \rangle$

$\vec{a} \cdot \vec{b} = 0 - 5 + 35 = \underline{30}$

$\|\vec{a}\| = \sqrt{9 + 1 + 49} = \sqrt{59}$

$\|\vec{b}\| = \sqrt{0 + 25 + 25} = \sqrt{50}$

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|}$

$\sin\theta = \dfrac{\|a \times b\|}{\|a\| \, \|b\|}$

6

Panel 7

Name:

Quiz #2

① Suppose $\vec{v} = \langle -1, -2, 3 \rangle$ and $\vec{w} = \langle 1, -4, -2 \rangle$

a) Find $\vec{v} \cdot \vec{w} = \langle -1, -2, 3 \rangle \cdot \langle 1, -4, -2 \rangle = -1 + 8 - 6 = 1$ ✓

$\|v\| = \sqrt{1+4+9} = \sqrt{14}$

$\|w\| = \sqrt{1+16+4} = \sqrt{21}$ ✓

b) Find the angle θ (or $\cos(\theta)$) between $\vec{v}$ and $\vec{w}$

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|v\|\|w\|} = \frac{1}{\sqrt{14}\sqrt{21}}$$ ✓

c) Find $\text{proj}_{\vec{v}}(\vec{w}) = \frac{v \cdot w}{\|v\|} \cdot \frac{v}{\|v\|} = \left(\frac{v \cdot w}{\|v\|^2}\right) v = \frac{1}{14}\langle -1, -2, 3 \rangle$

7

Panel 8

Quiz #2

② Suppose $\vec{v} = \langle -1, -2, 3 \rangle$ and $\vec{w} = \langle 1, -4, -2 \rangle$

a) Find $\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ -1 & -2 & 3 \\ 1 & -4 & -2 \end{vmatrix} = \langle 4 + 12, -(2 - 3), 4 + 2 \rangle$

$\langle 16, 1, 6 \rangle$

b) Find $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w} = \langle -16, -1, -6 \rangle$

c) Find $\vec{v} \cdot (\vec{w} \times \vec{v}) = 0$

8

Panel 9

Application: Torque

is an application of cross product!

"left for reading"

9

Panel 10

Of Lines and Planes

Line: $y = mx + b$ no good

1. Excludes vertical lines ✓
2. Does not generalize to higher dimension ✓

$z = mx + ny + b$ is not a line

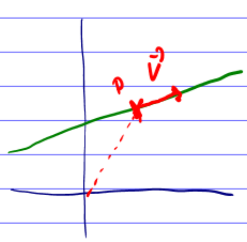
Use vectors for lines

line has direction: direction $\vec{v}$

and a fixed position: through point P_0

any point on line can be reached by

$$l(t) = P_0 + t\vec{v}$$



10

Panel 11

Def: If $P(x_0, y_0, z_0)$ is a point on a line, and $\vec{v} = \langle a, b, c \rangle$ is the direction of the line, then the (parametric) equation of the line is:

$$\ell(t) = P_0 + t\vec{v} \iff \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

in $\mathbb{R}^2$: $\ell(t) = (x_0, y_0) + t(a, b) = \langle x_0 + ta, y_0 + tb \rangle$

$x = x_0 + ta \Rightarrow t = \frac{1}{a}(x - x_0)$

$y = y_0 + tb$

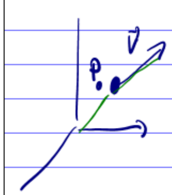
$y = y_0 + \frac{b}{a}(x - x_0)$ ← familiar equation

11

Panel 12

Ex: Find equation of a line

a) through $(5, 1, 3)$ and parallel to $\vec{v} = \langle 1, 4, -2 \rangle$



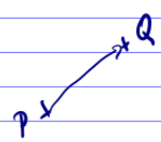
$\ell(t) = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle$

or $x = 5 + t$, $y = 1 + 4t$, $z = 3 - 2t$

b) through $P(1, 2, 3)$ and $Q(4, 1, 1)$

$\vec{PQ} = \langle 3, -1, -2 \rangle$ is direction of line

$\ell(t) = \langle 1, 2, 3 \rangle + t\langle 3, -1, -2 \rangle$



12

Panel 13

line through $(1,2)$ and $(3,4)$

old: ① $m = \frac{y_2 - y_1}{x_2 - x_1}$

② point-slope formula

new: $\ell(t) = \langle 1,2 \rangle + t \langle 2,2 \rangle$

13

Panel 14

Ex: At what point does $\langle 2,4,-4 \rangle + t\langle 1,-5,4 \rangle$ intersect the xy-plane

intersect xy-plane $\iff$ $\underline{z=0}$

$\ell(t) = \langle 2,4,-4 \rangle + t\langle 1,-5,4 \rangle \iff$

$x = 2+t \qquad x = 3$

$y = 4-5t \qquad y = -1$

$z = -4+4t = 0 \iff t = 1$

line intersects xy plane at $P(3,-1,0)$

14

Panel 15

Ex: Suppose 2 lines are $l_1(t) = \langle 1+t, -2+3t, 4-t \rangle$
$l_2(s) = \langle 2s, 3+s, -3+4s \rangle$

a) Are the lines parallel?

dir $v_1 = \langle 1, 3, -1 \rangle$
$v_2 = \langle 2, 1, 4 \rangle$ No because $v_1 \neq v_2$

b) Do they intersect in $\mathbb{R}^3$?

If they did: $l_1(t) = l_2(s)$

$1+t = 2s$ $s = ?$
$-2+3t = 3+s$ $t = ?$
$4-t = -3+4s$

15