Panel 1

Last Time

def Dot Product: $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Geometrically: $\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} = \cos(\theta)$

Projection of $\vec{b}$ onto $\vec{a}$: $proj_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\|a\|^2} \vec{a}$ memorize

Length of $proj_{\vec{a}}(\vec{b})$: $comp_{\vec{a}}(\vec{b}) = \frac{|\vec{a} \cdot \vec{b}|}{\|a\|}$

Properties of dot product: $\vec{a} \cdot \vec{a} = \|a\|^2$

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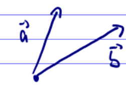
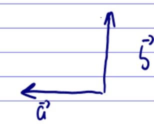
Panel 2

Picture Problems

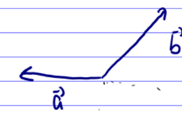
$\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} = \cos\theta \Leftrightarrow \vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos(\theta)$

(1) 

$\vec{a} \cdot \vec{b}$ positive / zero / negative

(2) 

$\vec{a} \cdot \vec{b}$ positive / zero / negative

(3) 

$\vec{a} \cdot \vec{b}$ positive / zero / negative

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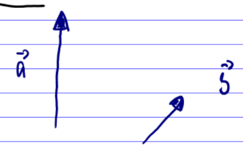
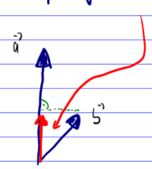
Panel 3

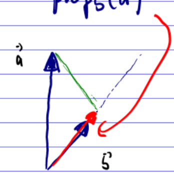
More picture Problems



find $proj_{\vec{a}}(\vec{b}) = \frac{a \cdot b}{\|a\|^2} \vec{a}$

$proj_{\vec{b}}(\vec{a}) = \frac{a \cdot b}{\|b\|^2} \vec{b}$





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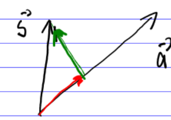
Panel 4



It looks like

$proj_a(\vec{b}) - \vec{b}$ ← is the green vector $\perp \vec{a}$

Prove it:

i.e. $(proj_{\vec{a}}(\vec{b}) - \vec{b}) \cdot \vec{a} = 0$?

$\left(\frac{a \cdot b}{\|a\|^2} \cdot \vec{a} - \vec{b}\right) \cdot \vec{a} =$

$\frac{a \cdot b}{\|a\|^2} \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a}$

$\frac{a \cdot b}{\cancel{\|a\|^2}} \cancel{\|a\|^2} - \vec{b} \cdot \vec{a} = \vec{a}\vec{b} - \vec{b}\vec{a} = 0$

$\vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}$

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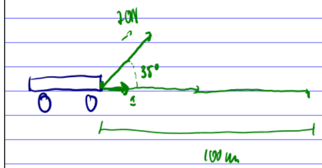
Panel 5

Application: A wagon is pulled a distance of 100 m by a constant force of 70 N, applied to a handle held at 35°. Find work done by $\vec F$.



$$comp_{\vec i}(\vec F) = \frac{\vec F \cdot \vec i}{\|\vec i\|} = \vec F \cdot \vec i = \|F\|\|i\|\cos(\theta)$$

$$= 70\cos(35)$$

$\Rightarrow$

work $W = 70N\cos(35) \cdot 100\,m = \underline{5734\ Nm}$

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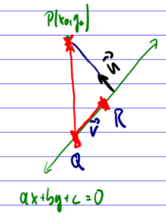
Panel 6

Find formula for distance of a point $P(x_0, y_0)$ from a line $ax + by + c = 0$.

dist: $\dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$



Need 2 points on line:

$Q(0, -\frac{c}{b})$, $R(-\frac{c}{a}, 0)$

$\Rightarrow QR = \langle -\frac{c}{a}, \frac{c}{b} \rangle \cdot \frac{ab}{c} \sim \vec v = \langle -b, a \rangle$

$\Rightarrow \vec n = \langle a, b \rangle$ is perpendicular to $\vec v = \langle -b, a \rangle$

$\Rightarrow$ dist $(P, \text{line}) = comp_{\vec n}(QP) = \dfrac{|\vec{QP} \cdot \vec n|}{\|\vec n\|} = \dfrac{\langle x_0, y_0 + \frac{c}{b}\rangle \cdot \langle a, b\rangle}{\sqrt{a^2+b^2}} = \underline{\dfrac{ax_0 + by_0 + c}{\sqrt{a^2+b^2}}}$

where $QP = \langle x_0, y_0 + \frac{c}{b} \rangle$

$\langle 1, 2 \rangle \cdot \langle -2, 1 \rangle$

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Panel 7

So: Add / Subtract vector $\to$ vector

Dot product of vectors $\to$ scalar

Cross Product: $\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle =$

$\langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$

good news: $\vec a \times \vec b$ is vector

bad news: only works in $\mathbb{R}^3$

weird!

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Panel 8

How to memorize the cross product:

$$\vec a \times \vec b = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

$$\vec a \times \vec b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle \overset{i}{a_2 b_3 - a_3 b_2},\ \overset{j}{-(a_1 b_3 - a_3 b_1)},\ \overset{k}{a_1 b_2 - a_2 b_1} \rangle$$

Ex: $\langle 1, 3, 4 \rangle \times \langle 2, 7, -5 \rangle = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} =$

$\langle -15 - 28,\ -(-5 - 8),\ 7 - 6 \rangle =$

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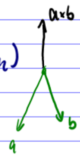
Panel 9

Properties: (1) $\vec{a} \times \vec{a} = \vec{0}$ $\qquad i \times j = k$

(2) $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

Proof of (1): $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \langle a_2a_3 - a_2a_3, -(a_1a_3 - a_1a_3), a_1a_2 - a_1a_2 \rangle = \vec{0}$



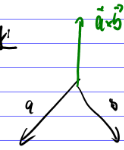
proof of (2): $\vec{a} \times \vec{b} \cdot \vec{a} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle =$

$= \cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_1a_2b_3} + \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1} =$

$= 0$

Panel 10

Geometry:



right-hand rule applies

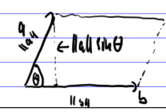
$\boxed{\|\vec{a} \times \vec{b}\| = \|a\| \|b\| \cdot \sin(\theta)}$ $\qquad$ recall $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$

Cor: $\vec{a}, \vec{b}$ are parallel iff $\vec{a} \times \vec{b} = \vec{0}$ overkill

$\vec{a}, \vec{b}$ are perpendicular iff $\vec{a} \cdot \vec{b} = 0$ useful

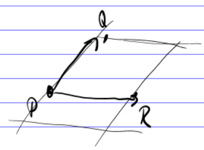
$\langle 1,2,3 \rangle$ and $\langle 4,8,12 \rangle$ are parallel because $\langle 4,8,12 \rangle = 4 \cdot \langle 1,2,3 \rangle$

Note: $\underbrace{\|a \times b\|}_{\|a\| \|b\| \sin\theta}$ is area of parallelogram spanned by $\vec{a}, \vec{b}$



Panel 11

Ex: Find vector perpendicular to the plane through $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$



answer: $\vec{PQ} \times \vec{PR} =$

$= \langle -3, 1, -7 \rangle \times \langle 0, -5, -5 \rangle = \ldots$ HW

Panel 12

Q: Is $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$? NOT:

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ anti commutative

as HW: find example

Panel 13

Cross out the expressions that do not make sense. For the rest, is the answer a vector or a scalar?

$a \cdot (b \times c)$ #	$\cancel{(a \cdot b) \times c}$
$\Rightarrow$ $\cancel{(a \cdot b) \cdot c}$	$(a \times b) + c$ ✓
$\cancel{a \times (b \cdot c)}$	$\cancel{(a \cdot b) \times (c \cdot d)}$
$\lVert a \rVert (b \cdot c)$ #	$-\cancel{(a \cdot b) + c}$
$a \times (b \times c)$ vect	$\Rightarrow$ $(a \times b) \cdot (c \times d)$ ✓
$a \cdot (b + c)$ #	$\lVert a \rVert (b \times c)$ ✓
$(a \times b) + c$ vect.	$\lVert a \times b \rVert$ ✓
	$\cancel{\lVert a \cdot b \rVert}$ ~ $\lvert a \cdot b \rvert$

Quiz Monday

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