Maple – Optional

We want to investigate in which dimension the unit ball has the largest "volume". Recall:

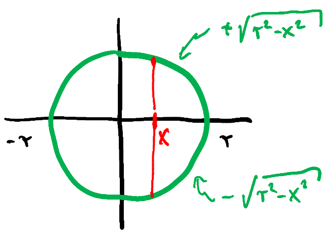
- 1-D "ball": $B_1(r) = \{x \in \boldsymbol{R}: x^2 < r^2\}$ (which would be the interval $(-r, r)$)
- 2-D "ball": $B_2(r) = \{(x, y) \in \boldsymbol{R^2}: x^2 + y^2 < r^2\}$ (also known as the disk of radius *r*)
- 3-D "ball": $B_3(r) = \{(x, y, z) \in \boldsymbol{R^3}: x^2 + y^2 + z^2 < r^2\}$ (a standard 3D ball of radius *r*)r
- 4-D "ball": $B_4(r) = \{(x, y, z, w) \in \boldsymbol{R^4}: x^2 + y^2 + z^2 + w^2 < r^2\}$
- …
- n-D "ball": $B_n(r) = \{(x_1, x_2, \dots, x_n) \in \boldsymbol{R^n}: x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 < r^2\}$

Drawing pictures of the 2D and 3D cases to determine the bounds, we know that:

- "volume" of $B_1(r)$:
  - $vol(B_1(r)) = \int_{B_1} dx = \int_{-r}^{r} dx = 2r$ (length of the interval (-r,r))
  - Maple: $int(1, x = -r..r);$

$$2\, r$$

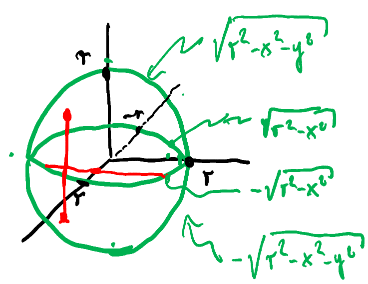
- "volume" of $B_2(r)$:



  - $vol(B_2(r)) = \iint_{B_2(r)} dA = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy\, dx = \pi r^2$ (area of disk radius r)
  - Maple: $int(int(1, y = -\mathrm{sqrt}(r^2 - x^2)..\mathrm{sqrt}(r^2 - x^2)), x = -r..r)$

$$\mathrm{csgn}(r)\, r^2\, \pi$$

- Volume of $B_3(r)$:



  - $vol(B_3(r)) = \int\int\int_{B_3(r)} dV = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} dz\, dy\, dx = \frac{4}{3}\pi r^3$
  - Maple: $int(int(int(1, z = -\mathrm{sqrt}(r^2 - x^2 - y^2)..\mathrm{sqrt}(r^2 - x^2 - y^2)), y = -\mathrm{sqrt}(r^2 - x^2)..\mathrm{sqrt}(r^2 - x^2)), x = -r..r)$

$$\frac{4}{3}\,\pi\, r^3$$

I have used Maple to evaluate these integrals for us. Of course we could have also used polar coordinates, but Maple gives us the answers just fine. Moreover, there is a simple pattern here, so it should be easy to verify, using Maple, that, for example:

- Volume of $B_4(r) = \frac{1}{2}\pi^2 r^4$

Your Assignment:

1. Compute the volumes of the 1D, 2D, 3D, …, 10D balls of radius r
2. Substitute *r = 1* in the volumes found in step 1 and get a decimal answer for each n-ball
3. Which of these unit balls has the largest volume, and what is it?

This would be good enough for this assignment. It is due on the day of our final exam, either via email or in person – no exceptions.

Technically, though, this is no proof since we only consider the first 10 dimensions. It turns out, however, that the “true” maximum volume of the unit n-ball occurs at a fractional dimension (whatever that is), the closest integer of which should be *your* answer.

For extra Extra Credit:

Let’s look at “volume” and “area” of 1D, 2D, and 3D balls:

	“Volume”	**“Area” of boundary**
1D ball (interval)	$2\ r$ (length of interval)	0 (2 points)
2D ball (unit disk)	πr^2 (standard area)	$2\pi\ r$ (length of circle)
3D ball	$\frac{4}{3}\pi\ r^3$ (see above)	$4\pi\ r^2$ (surface area of sphere)
4D ball	$\frac{1}{2}\pi^2 r^4$ (see above)	$2\pi^2 r^3$ (verify)

1. There seems to be an interesting relationship of the “volume” to the “surface area” of 2D, 3D, and 4D balls. What is it (consider r as the variable)?
2. Compute the “surface area” of the remaining balls for which you know the volume, using Maple to do the computations.
3. Does the relation you conjectured remain true in all of your examples?
4. Prove (or find a counterexample for) your conjecture (not so easy).