Math 2511 – Calc III Practice Exam 1

This is a practice exam. The actual exam consists of questions of the type found in this practice exam, but will be shorter. If you have questions do not hesitate to send me email.

1. **Definitions**: Please state in your own words the meaning of the following terms:
   a) Vector
   b) Angle between two vectors
   c) Unit vector
   d) Tangent vector to a curve
   e) Unit tangent vector to a curve
   f) Normal vector to a curve
   g) Binormal vector
   h) Curvature
   i) Length of a curve
   j) Velocity, speed, and acceleration
   k) Tangential component of acceleration
   l) Normal component of the acceleration

2. **True/False** questions:
   a) $\boldsymbol{u}\cdot\boldsymbol{u} = \|\boldsymbol{u}\|^2$ T

   b) $<1,3,2>$ and $<-4,-2,5>$ are perpendicular T

   c) $<1,3,-2>$ and $<2,6,4>$ are parallel F

   d) $\boldsymbol{v}\cdot\boldsymbol{w} = -\boldsymbol{w}\cdot\boldsymbol{v}$ F

   e) $\dfrac{d}{dt}\|\boldsymbol{r}(t)\| = \left\|\dfrac{d}{dt}\boldsymbol{r}(t)\right\|$ F

   f) $\dfrac{d}{dt}\boldsymbol{p}(t)\times\boldsymbol{r}(t) = \boldsymbol{p}'(t)\times\boldsymbol{r}'(t)$ F

   g) $\boldsymbol{r}(t) = \left\langle \sqrt{t}+2, 3-\sqrt[3]{t}, \sqrt[4]{t}\right\rangle$ is the equation of a line F

   h) If $\|\boldsymbol{r}(t)\| \equiv 1$ then $\boldsymbol{r}(t)\times\boldsymbol{r}'(t) = 0$ T

   i) The planes $x+3y+2z=5$ and $4x+2y-5z=0$ are perpendicular T

   j) The distance between $x-y+z=2$ and $x+y+z=1$ is zero T

3. **Vectors**: Suppose $\boldsymbol{u} = \langle 7,-2,3 \rangle$, $\boldsymbol{v} = \langle -1,4,5 \rangle$, and $\boldsymbol{w} = \langle -2,1,-3 \rangle$
   a) Are u and v orthogonal, parallel, or neither?

   $\langle 7,-2,3\rangle \cdot \langle -1,4,5\rangle = 0$ so perpendicular (= orthogonal)

   b) Find graphically and algebraically $2\boldsymbol{u}+3\boldsymbol{v}$ and $\boldsymbol{u}-\boldsymbol{v}$

   easy. – can you do it graphically?

   c) Find the angle between v and w

   $$\cos(\alpha) = \frac{v \cdot w}{\|v\| \cdot \|w\|} = \frac{\langle -1,4,5\rangle \cdot \langle -2,1,-3\rangle}{\sqrt{42}\,\sqrt{14}} = \frac{2+4-15}{\sqrt{42}\,\sqrt{14}} = \underline{\underline{-9/\sqrt{42}\sqrt{14}}}$$

   d) Find $\boldsymbol{u}\cdot\boldsymbol{v}$ (dot product), $\boldsymbol{u}\times\boldsymbol{v}$ (cross product), $\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})$, and $\|\boldsymbol{u}\|$

   let's do cross product only!

   $u \times v$

   $$\begin{vmatrix} i & j & k \\ 7 & -2 & 3 \\ -1 & 4 & 5 \end{vmatrix} = \langle -10-12, -(35+3), 28-2 \rangle = \underline{\underline{\langle -22, -38, 26 \rangle}}$$

   e) Find the projection of w onto u and the projection of u onto w

   $$\text{proj}_{\vec{u}}(\vec{w}) = \frac{u \cdot w}{\|u\|^2}\vec{u} = \frac{\langle 7,-2,3\rangle \cdot \langle -2,1,-3\rangle}{(\sqrt{62})^2}\vec{u} = \underline{\underline{\frac{-14-2-9}{62}}} \cdot \vec{u} =$$

4. **Lines and Planes**
   a) Find the equation of the plane spanned by $\langle 1,3,-2 \rangle$ and $\langle 2,1,2 \rangle$ through the point $P(1,2,3)$

   $$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 3 & -2 \\ 2 & 1 & 2 \end{vmatrix} = \_\_$$

   then $\_\_(x-1) + \_\_(y-2) + \_\_(z-3) = 0$

b) Find the equation of the plane through $P(1,2,3)$, $Q(1,-1,1)$, and $R(3,2,1)$

① Find $\vec{PQ}$

② Find $\vec{PR}$

③ Find $\vec{PQ} \times \vec{PR}$

$\Rightarrow \ \_\_(x-1) + \_\_(y-2) + \_\_(z-3) = 0$

c) Find the equation of the plane parallel to $x - y + z = 2$ through $P(0,2,0)$

$\vec{n} = \langle 1,-1,1 \rangle$

$\Rightarrow (x-0) - (y-2) + (z-0) = 0$ or $\underline{x - y + z = -2}$

d) Find the equation of the line through $P(1,2,3)$ and $Q(1,-1,1)$

$l(t) = (1,2,3) + t \cdot \vec{PQ}$

e) Do the plane $x - y + z = 2$ and the line $l(t) = \langle 1+t, 2t, 1-5t \rangle$ intersect? If so, where?

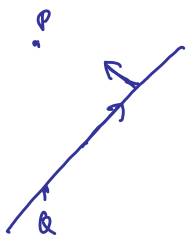
$l(t) = \langle x(t), y(t), z(t) \rangle = \langle \underset{x}{1+t}, \underset{y}{2t}, \underset{z}{1-5t} \rangle$. To be on the plane we

need $x - y + z = 2$ or $(1+t) + (2t) + (1-5t) = 2 \Rightarrow t = 0$

$\Rightarrow l(0) = \underline{\langle 1,0,1 \rangle}$ is point of intersection

5. **Distances**

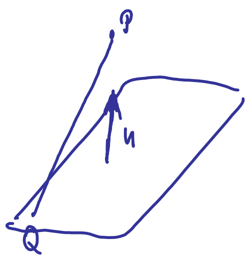
a) Find the distance between the line $x - y = 2$ and $P(1,2)$



$\vec{n} = \langle 1,-1 \rangle$, $Q = (2,0)$ for example

$d = \text{proj}_{\vec{n}} \vec{PQ} = \dfrac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} = \underline{\dfrac{\vec{PQ} \cdot \langle 1,-1 \rangle}{\sqrt{2}}}$

b) Find the distance between the plane $x + y + z = 1$ and the point $P(1, 2, 3)$



$\vec{n} = \langle 1,1,1 \rangle$, $Q = (1,0,0)$ for example

$$\Rightarrow d = \frac{\vec{PQ} \cdot \langle 1,1,1 \rangle}{\sqrt{3}}$$

c) Find the distance between the planes $x - y + z = 2$ and $2x - 2y + 2z = 5$

$n_1 = \langle 1,-1,1 \rangle$ $\quad n_2 = \langle 2,-2,2 \rangle$

so planes are parallel, i.e. distance is not zero.

$P \in$ plane 1 : $P(2,0,0)$

$Q \in$ plane 2 : $Q(\tfrac{5}{2},0,0)$

$$\Rightarrow d = \frac{\vec{PQ} \cdot \langle 2,-2,2 \rangle}{\sqrt{12}}$$

6. **Vector valued functions**:

a) Find $r'(t)$ if $r(t) = < 6t, -7t^2, t^3 >$

$r'(t) = \langle 6, -14t, 3t^2 \rangle$

b) Find $r'(t)$ if $r(t) = < a\cos^3(t), a\sin^3(t), t\sin(t) >$

$r'(t) = \langle -3a\cos^2(t)\sin(t), 3a\sin^2(t)\cos(t), \sin(t) + t\cos(t) \rangle$

c) If $r(t) = < 4t, t^2, t^3 >$, find $r'(t)$, $r''(t)$, $\frac{d}{dt}\|r(t)\|$

$r'(t) = \langle 4, 2t, 3t^2 \rangle$

$r''(t) = \langle 0, 2, 6t \rangle$

$\|r(t)\| = \sqrt{16t^2 + t^4 + t^6}$

$$\Rightarrow \frac{d}{dt}\|r(t)\| = \frac{1}{2}\left(16t^2 + t^4 + t^6\right)^{-1/2} \cdot \left(32t + 4t^3 + 6t^5\right)$$

d) If $\boldsymbol{r}(t) = < e^t, 3t^3, \frac{3}{6t} >$ some curve, find $\int_1^2 \boldsymbol{r}(t)dt$

Straight-forward

$r'(t) = \langle 1, -1/t^2 \rangle$, $\|r'(t)\| = \sqrt{1 + 1/t^4}$

e) If $\boldsymbol{r}(t) = < t, \frac{1}{t} >$, find $T(t)$, $N(t)$, a_t and a_n

$\Rightarrow T(t) = \frac{1}{\sqrt{1 + 1/t^4}} \langle 1, -1/t^2 \rangle$

The normal is a unit vector perp. to T. In 2D this is easy to find

$$\underline{N(t) = \frac{1}{\sqrt{1 + 1/t^4}} \langle 1/t^2, 1 \rangle}$$

$a_T = \frac{v \cdot a}{s}$, $a_N = \frac{\|v \times a\|}{s}$ $\rightarrow$ left as HW

f) Repeat (e) for $\boldsymbol{r}(t) = < e^t \cos(t), e^t \sin(t) >$ for $t = \frac{\pi}{2}$

$r'(t) = \langle e^t \cos(t) - e^t \sin(t), e^t \sin(t) + e^t \cos(t) \rangle = e^t \langle \cos(t) - \sin(t), \cos(t) + \sin(t) \rangle$

$\|r'(t)\| = e^t \sqrt{(\cos(t) - \sin(t))^2 + (\cos(t) + \sin(t))^2} =$

$= e^t \sqrt{\cos^2 - \cancel{2\sin\cos} + \sin^2 + \cos^2 + \cancel{2\cos\sin} + \sin^2} = \sqrt{2} e^t$

$\Rightarrow T(t) = \frac{1}{\sqrt{2}} \langle \cos(t) - \sin(t), \cos(t) + \sin(t) \rangle$ $\Rightarrow$ $N(t) = \frac{1}{\sqrt{2}} \langle \cos(t) + \sin(t), \sin(t) - \cos(t) \rangle$, a_T, a_N too much

g) If $\boldsymbol{r}(t) = < 3 - 3t, 4t >$, find the arc length of the curve between 0 and 1

$$S = \int_0^1 \|r'\| dt = \int_0^1 \| \langle -3, 4 \rangle \| dt = \underline{5}$$

h) If $\boldsymbol{r}(t) = < 4t, 3\cos(t), 3\sin(t) >$, find the arc length of the curve between 0 and $\frac{\pi}{2}$

$$S = \int_0^{\pi/2} \|r'\| dt = \int_0^{\pi/2} \| \langle 4, -3\sin(t), 3\cos(t) \rangle \| = \underline{5 \cdot \pi/2}$$

i) Find the curvature of $\boldsymbol{r}(t) = < t, 3t^2, \frac{t^2}{2} >$

$r' = \langle 1, 6t, t \rangle$

$r'' = \langle 0, 6, 1 \rangle$

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\sqrt{37}}{\left(1+37t^2\right)^{3/2}}$$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & 6t & t \\ 0 & 6 & 1 \end{vmatrix} = \langle 6t - 6t, -(1), 6 \rangle = \langle 0, 1, 6 \rangle$$

7. **Motion in space:** In each of the following problems, $\boldsymbol{r}(t)$ represents the position vector of particle in space at time t.

a) If $\boldsymbol{r}(t) = < t, 3t^2, \frac{t^2}{2} >$, find the velocity, speed, and acceleration.

$v = r' = \langle 1, 6t, t \rangle$, $s = \|v\| = \sqrt{1+37t^2}$

$a = r'' = \langle 0, 6, 1 \rangle$

b) If $\boldsymbol{r}(t) = < t, 3t^2, \frac{t^2}{2} >$, find tangential and normal components of the acceleration

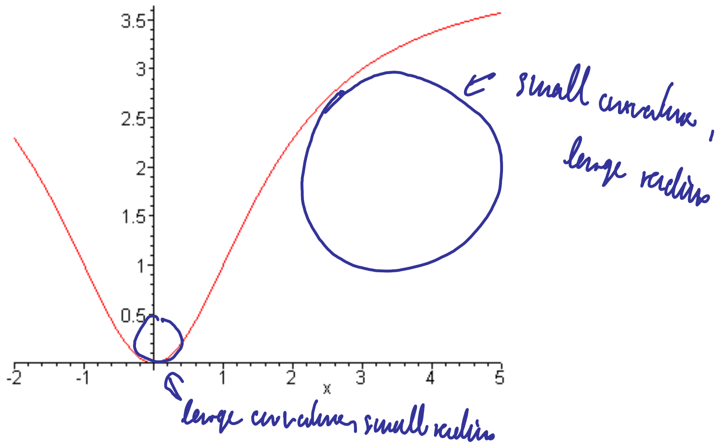
$v(t) = \langle 1, 6t, t \rangle$, $s = \sqrt{1+37t^2}$

$v''(t) = \langle 0, 6, 1 \rangle$

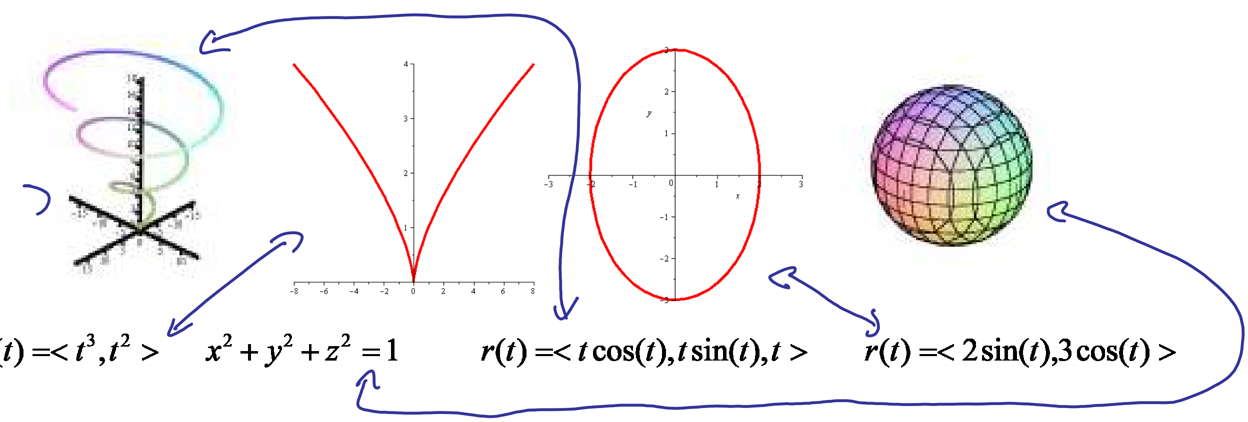
$$a_T = \frac{v \cdot a}{s} = \frac{\langle 1, 6t, t \rangle \cdot \langle 0, 6, 1 \rangle}{\sqrt{1+37t^2}} = \frac{37t}{\sqrt{1+37t^2}}$$

$$a_N = \frac{\|v \times a\|}{s} = \frac{\|\langle 0, 1, 6 \rangle\|}{\sqrt{1+37t^2}} = \frac{\sqrt{37}}{\sqrt{1+37t^2}}$$

8. **Picture:** Sketch the circle that fits the graph below the best at the points x = 0 and x = 3. At which of the two points is the curvature smaller?

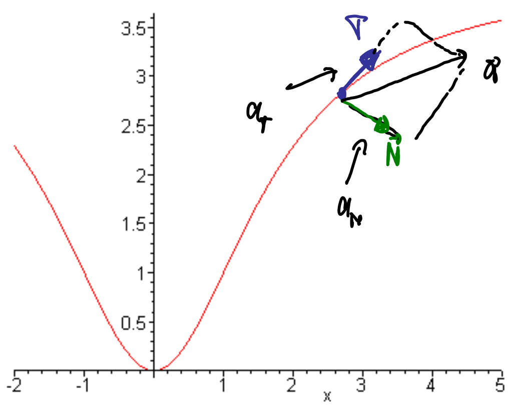


9. **Picture**: Match the following functions to their corresponding plots.



$r(t) = < t^3, t^2 >$ $\quad x^2 + y^2 + z^2 = 1$ $\quad r(t) = < t\cos(t), t\sin(t), t >$ $\quad r(t) = < 2\sin(t), 3\cos(t) >$

10. **Picture**: The graph below shows a vector-valued function. Sketch the unit tangent, unit normal, acceleration, tangential and normal components of the acceleration for $t = 3$.



11. **Story problem** (motion)

a) A baseball is hit 3 feet above ground at 100 feet per second and at an angle of Pi/4 with respect to the ground. Find the maximum height reached by the baseball. Will it clear a 10-foot high fence located 300 feet from home base?

max height is done like last questions

Know: $r(0) = (0, 3)$

$r'(0) = \frac{100}{\sqrt2} \langle 1, 1 \rangle$

$r''(t) = \langle 0, -g \rangle$

$\Rightarrow r'(t) = \langle c_1, -gt + c_2 \rangle = \langle \frac{100}{\sqrt2}, -gt + \frac{100}{\sqrt2} \rangle$

$\Rightarrow r(t) = \langle \frac{100}{\sqrt2} t + d_1, -\frac12 g t^2 + \frac{100}{\sqrt2} t + d_1 \rangle = \langle \frac{100}{\sqrt2} t, -\frac12 g t^2 + \frac{100}{\sqrt2} t + 3 \rangle$

If $x = 300 \Rightarrow \frac{100}{\sqrt2} t = 300 \Rightarrow t = 3\sqrt2 \Rightarrow$ height $y(3\sqrt2) = -\frac12 32 \cdot 18 + \frac{100}{\sqrt2} \cdot \sqrt2 \cdot 3 + 3 = 14 > 10$ so ball clears wall

b) What is the maximum height and range of a projectile fired at a height of 3 feet above the ground with an initial velocity of 900 feet/sec and at an angle of 45 degrees above the horizontal?

See below

11. **Prove** the following facts:

a) Show that $\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$

easy

b) Show that $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{u}) = \boldsymbol{0}$

easy - do this

c) Show that if $y = f(x)$ is a function that is twice continuously differentiable, then the curvature of f at a point x is $K = \dfrac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{3/2}}$

$y = f(x) \Leftrightarrow r(t) = \langle t, f(t) \rangle = \langle t, f(t), 0 \rangle$

$r(t) = \langle t, f(t), 0 \rangle$

$r'(t) = \langle 1, f'(t), 0 \rangle$

$r''(t) = \langle 0, f''(t), 0 \rangle$

$\Rightarrow r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & f' & 0 \\ 0 & f'' & 0 \end{vmatrix} = \langle 0, 0, f'' \rangle$

$\Rightarrow \chi = \dfrac{\|r' \times r''\|}{\|r'\|^3} = \dfrac{|f''(t)|}{(1 + (f'(t))^2)^{3/2}}$ qed.

d) Prove that the curvature of a line in space is zero.

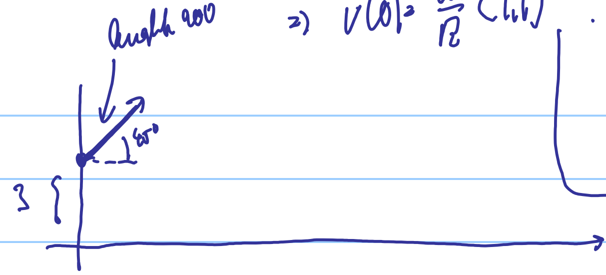
$\ell(t) = \langle a_1 t + b_1, a_2 t + b_2, a_3 t + b_3 \rangle$ is a general line.

$\ell'(t) = \langle a_1, a_2, a_3 \rangle$

$\ell''(t) = \langle 0, 0, 0 \rangle$

$\Rightarrow \chi = \dfrac{\ell' \times \ell''}{\|\ell'\|^3} = 0$

1(b)



$\Rightarrow \vec{v}(0) = \frac{900}{\sqrt{2}}\langle 1,1\rangle$. Also $r(0) = \langle 0,3\rangle$ and

$a(t) = \langle 0, -g\rangle$

$\Rightarrow v(t) = \langle \frac{900}{\sqrt{2}}, -gt + 900/\sqrt{2}\rangle$

$r(t) = \langle \frac{900}{\sqrt{2}}t, -\frac{1}{2}gt^2 + \frac{900}{\sqrt{2}}t + 3\rangle$

Max height: $y(t) = -\frac{1}{2}gt^2 + \frac{900}{\sqrt{2}}t + 3 \Rightarrow y' = -gt + \frac{900}{\sqrt{2}} = 0 \Rightarrow t = \underline{\underline{\frac{900}{g\sqrt{2}}}}$ is critical and gives a max

$\Rightarrow$ max distance: $x\left(\frac{900}{\sqrt{2}g}\right) = \frac{900}{\sqrt{2}} \cdot \frac{900}{\sqrt{2}g} = \underline{\underline{\frac{810000}{2g}}}$ feet