Panel 1

Review (and Correction)

If $f(x,y)$ is a function, $\gamma(t) = \langle x(t), y(t)\rangle$ a curve then

$$s = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt \quad \text{is length of curve}$$

$$\frac{ds}{dt} = \sqrt{(x')^2 + (y')^2}$$

$$\Rightarrow ds = \sqrt{\quad}\, dt$$

$$\int_\gamma f(x,y)\, \cancel{dr}\, ds = \int_a^b f(x(t), y(t))\, \sqrt{(x')^2 + (y')^2}\, dt$$

$$\int_\gamma f(x,y)\, dx = \int_a^b f(x(t), y(t))\, \frac{dx}{dt}\, dt = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_\gamma f(x,y)\, dy = \int_a^b f(x(t), y(t))\, \frac{dy}{dt}\, dt = \int f(x(t), y(t))\, y'(t)\, dt$$

If $F = \langle M, N\rangle$: $\displaystyle\int_\gamma \vec{F}\cdot d\vec{r} = \int_\gamma M\,dx + N\,dy \quad (dr = \langle dx, dy\rangle)$

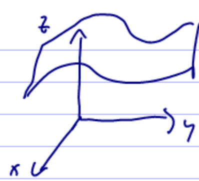
1

Panel 2

Let $f(x,y) = x^2 - xy + y^2$, $F(x,y) = \langle 2x - y, 2y - x\rangle$, $D = \{(x,y) : x^2 + y^2 \le 1\}$, $C = \{(x,y) : x^2 + y^2 = 1, y \ge 0\}$, $\gamma_1(t) = \langle t, 0\rangle$, $t \in [-1,1]$, and $\gamma_2(t) = \langle t, \sin(\pi t)\rangle$, $t \in [-1,1]$.

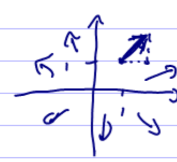
1. Sketch each object

$z = f(x,y) = x^2 - xy + y^2$ is surface in $\mathbb{R}^3$



$plot3d(x^2 - x\cdot y + y^2, x = -4..4, y = -4..4)$

$F(x,y) = \langle 2x - y, 2y - x\rangle$ is vector field in $\mathbb{R}^2$



$fieldplot(\langle 2\cdot x - y, 2\cdot y - x\rangle, x = -3..3, y = -3..3)$

$D = \{(x,y) : x^2 + y^2 \le 1\}$ set in $\mathbb{R}^2$

No Maple inside unit circle

$C = \{(x,y) : x^2 + y^2 = 1, y \ge 0\}$ curve in $\mathbb{R}^2$

$implicitplot(x^2 + y^2 = 1, x = -2..2, y = -2..2)$ (circle)

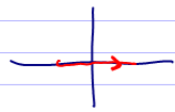
2

Panel 3

Let $f(x,y)=x^2-xy+y^2$, $F(x,y)=<2x-y,2y-x>$, $D=\{(x,y): x^2+y^2\le 1\}$,
$C=\{(x,y): x^2+y^2=1, y\ge 0\}$, $\gamma_1(t)=<t,0>$, $t\in[-1,1]$, and $\gamma_2(t)=<t,\sin(\pi t)>$, $t\in[-1,1]$.

1. Sketch each object

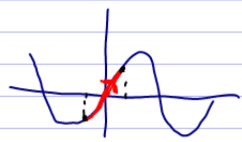
$\gamma_1(t) = \langle t, 0\rangle$, $t\in[-1,1]$ (parametrized) curve, incl. dir



$\gamma_2(t) = \langle t, \sin(t)\rangle$, $t\in[-1,1]$ (parametrized) curve with dir

plot([t, sin(t), t=-1..1]);

-1 sin(t)



3

Panel 4

Let $f(x,y)=x^2-xy+y^2$, $F(x,y)=<\overbrace{2x-y}^{M},\overbrace{2y-x}^{N}>$, $D=\{(x,y): x^2+y^2\le 1\}$,
$C=\{(x,y): x^2+y^2=1, y\ge 0\}$, $\gamma_1(t)=<t,0>$, $t\in[-1,1]$, and $\gamma_2(t)=<t,\sin(\pi t)>$, $t\in[-1,1]$.

a) ~~$\int_D f(x,y)dA$~~ $\int_D$ no good ($\iint_D$ okay)

b) ~~$\int_D f(x,y)ds$~~

✓ c) $\int_C f(x,y)ds$ = need $r(t) = \langle \overset{x}{\cos(t)}, \overset{y}{\sin(t)}\rangle$, t from 0 to π

$\int_C x^2-xy+y^2\, ds = \int_0^\pi [\cos^2(t)-\cos(t)\cdot\sin(t)+\sin^2(t)]\sqrt{(-\sin)^2+\cos^2(t)}\,dt$

✓ d) $\int_{\gamma_1} f(x,y)dx$

✓ e) $\int_{\gamma_2} f(x,y)dy$

f) ~~$\int_{\gamma_1} f(x,y)dr$~~ $dr = \langle dx, dy\rangle$, f dr no good

g) ~~$\int_{\gamma_1} F(x,y)dx$~~ F dx no good

h) ~~$\int_D F(x,y)dr$~~ can't have single integral over D

i) $\int_C F(x,y)dr$ ✓ $r(t)=\langle\cos(t),\sin(t)\rangle$, $t\in[0,\pi]$

j) $\int_{\gamma_1} F(x,y)dr$ ✓ $\int F\,dr = \int\langle M,N\rangle\cdot\langle dx,dy\rangle = \int M\,dx + \int N\,dy$

k) $\int_{\gamma_2} F(x,y)dr$ ✓ $= \int (2\cos(t)-\sin(t))(-\sin(t))dt + \int(2\sin(t)-\cos(t))\cos(t)\,dt$

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Panel 5

$$f(x,y) = x^2 - xy + y^2 \qquad \Rightarrow \gamma_1 = \langle t, 0 \rangle,\ t \in [-1,1]$$

$$\Rightarrow \gamma_2 = \langle t, \sin(t) \rangle,\ t \in [-1,1]$$

$$\int_{\gamma_1} f(x,y)\,dx = \int_{\gamma_1} x^2 - xy + y^2\,dx = \int t^2 \frac{dx}{dt} \cdot dt = \int_{-1}^{1} t^2 \cdot 1 \cdot dt$$

$$\int_{\gamma_2} f(x,y)\,dy = \int \left(t^2 - t\sin(t) + \sin^2(t)\right) \frac{dy}{dt}\,dt =$$

$$= \int_{-1}^{1} \left(t^2 - t\sin(t) + \sin^2(t)\right) \cdot \cos(t)\,dt \checkmark$$

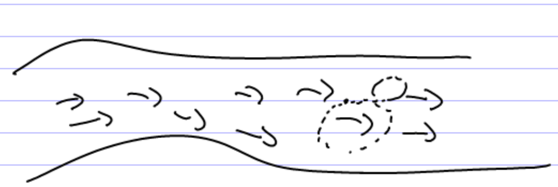
5

Panel 6

Note: $\int_C \vec{F}\, d\vec{\sigma}$ is important

because it gives "work":

Work of moving something around path C in a field $\vec{F}$.



Need

Short cut

6

Panel 7

Fundamental Theorem for Line Integrals $\left(\int_a^b f(x)\,dx = F(b) - F(a) \text{ in } \mathbb{R} \right)$

If $\vec{F}$ is conservative with potential function f, and $r(t)$, $a \le t \le b$, a smooth curve. Then:

$$\int_\gamma \vec{F}\,d\vec{r} = f(\underbrace{r(b)}_{\text{end}}) - f(\underbrace{r(a)}_{\text{starts}})$$

potential at end − potential at start

How to tell: $F(x,y) = \langle M, N \rangle$ conservative?

F is conservative if $\nabla f = F$, i.e.

$f_x = M$ and $f_y = N$

$\Rightarrow M_y = f_{xy}$ $\quad N_x = f_{yx}$ $\quad (f_{xy} = f_{yx})$

Conservative if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

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Panel 8

Which of the following vector fields is not conservative

(a) $F(x,y) = \langle x, y \rangle$ conservative

(b) $F(x,y) = \langle x^2+y^2, 2xy \rangle$ conserv.

conserv. (c) $F(x,y) = \langle e^x\cos(y), -e^x\sin(y) \rangle$ $\quad \frac{\partial}{\partial y}(e^x\cos(y)) = -e^x\sin(y)$, $\quad \frac{\partial}{\partial x}(-e^x\sin(y)) = -e^x\sin(y)$

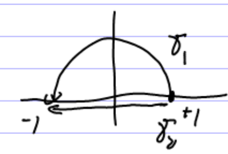
not conserv. (d) $F(x,y) = \langle x^2\cos(y), -y^2\sin(x) \rangle$

$\dfrac{\partial M}{\partial y} \overset{?}{=} \dfrac{\partial N}{\partial x}$ $\quad \left| \quad \begin{aligned} &\frac{\partial}{\partial y}\left(x^2\cos(y)\right) = -x^2\sin(y) \\ &\frac{\partial}{\partial x}\left(-y^2\sin(x)\right) = -y^2\cos(x) \end{aligned} \right\}$ not equal

8

Panel 9

Find $\int_{\gamma_1} \vec{F}\, d\vec{r}$ where $F(x,y) = \langle x^2+y^2, 2xy \rangle$ and $\gamma_1 \Rightarrow$



$\int_{\gamma_1} \vec{F}\, d\vec{r}$ ✓ and $\int_{\gamma_2} \vec{F}\, d\vec{r}$ ✓

<u>Old way:</u> $\int_{\gamma_1} F\, dr = \int_{\gamma_1} \underbrace{\langle x^2+y^2, 2xy\rangle}_{F} \cdot \underbrace{\langle dx, dy \rangle}_{dr} =$

$\gamma_1 : \langle \overset{x}{\cos(t)}, \overset{y}{\sin(t)} \rangle$, $t = 0$ to π

$= \int_{\gamma_1} x^2+y^2\, dx + \int_{\gamma_1} 2xy\, dy =$

$= \int_0^\pi [\cos^2(t) + \sin^2(t)](-\sin(t))dt + \int_0^\pi 2\cos(t)\sin(t)\cos(t)dt$

use Maple $\longrightarrow$ $= \int -\sin(t)\,dt + \int 2\sin(t)\cos^2(t)\,dt = -2/3$

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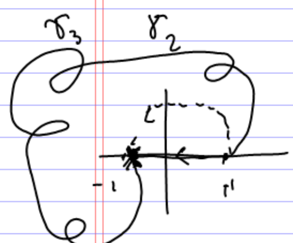
Panel 10

$\int_{\gamma_2} F\, dr = \int_{\gamma_2} x^2+y^2\, dx + 2xy\, dy = \int_1^{-1} t^2\, dt = -\frac{2}{3}$



$\gamma_2(t) = \langle \overset{x}{t}, \overset{y}{0} \rangle$, $t = 1$ to -1

<u>New way:</u> Is $F(x,y) = \langle x^2+y^2, 2xy \rangle$ conservative?

Potential function is $f(x,y) = \frac{1}{3}x^3 + xy^2$ (I guessed)

Now: $\int_{\gamma} F\, dr = f(\text{end}) - f(\text{start}) =$

$= f(-1,0) - f(1,0) =$

$= -1/3 - 1/3 = -2/3$

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Panel 11

Consequences of Fundamental Theorem for Line Integrals

Path Independence: If C_1 is a curve from A to B, and C_2 is another curve from A to B, and if F is conservative vector field, then

$$\int_{C_1} \vec{F}\, d\vec{r} = \int_{C_2} \vec{F}\, d\vec{r}$$, i.e.

Line integral is indep. of curve from A to B

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Panel 12

Ex: Find work done by gravitational field $\vec{F}(\vec{r}) = -\frac{mMG}{\|\vec{r}\|^3}\vec{r}$ moving particle from $(3,4,12)$ to $(2,2,0)$.

$F(x,y,z) = -\frac{mMG}{(x^2+y^2+z^2)^{3/2}} \cdot \langle x,y,z \rangle$ is conservative

with potential function $+mMG\,(x^2+y^2+z^2)^{-1/2} = f$

$$\int_{(3,4,12)}^{(2,2,0)} \vec{F}\, d\vec{r} = f(2,2,0) - f(3,4,12) = mMG\left[\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{169}}\right]$$

only makes sense if F is conservative!

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Panel 13

<u>Corollary 2:</u> If $\vec{F}$ is conservative and C a closed curve then $\int_C \vec{F}\,d\vec{r} = 0$

Note: if C is closed we write $\oint_C \vec{F}\,d\vec{r}$

Because F conserv., closed curve has same start/end point.

$$\Rightarrow \int_C F\,dr = \underbrace{f(\text{end}) - f(\text{start})}_{\text{same}} = 0$$

think Physics

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Panel 14

<u>How to find a Potential Function</u>

$F(x,y) = \langle M, N \rangle$ a vector field.

Want: $f(x,y)$ s.t. $\nabla f = F$ (if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$)

integr. ① $f_x = M \Rightarrow f = \text{"}\int M\,dx\text{"}$ (anti-deriv. with x)

f will include a function of y, $C(y)$

diff. ② Use f in ① and compute f_y. Compare

$f_y = \ldots + C'(y) = N$

solve ③ Solve equation ② for $C(y)$ by integration

check ④ Check your answer

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Panel 15

Find potential function for $\vec{F} = (\underbrace{3+2xy}_{M}, \underbrace{x^2-3y^2}_{N})$ if exists

integr ① $f_x = 3+2xy$, $f = \int 3+2xy \, dx$

$f = 3x + x^2y + C(y)$

diff ② $f_y = x^2 + C'(y) = x^2 - 3y^2$

$C'(y) = -3y^2$

solve ③ $C(y) = -y^3 + c$

④ $f(x,y) = 3x + x^2y - y^3 + C$

check

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Panel 16

Find potential function for $\langle x^2\cos(y), -y^2\sin(x) \rangle$

① $\frac{\partial}{\partial x} f = x^2\cos(y)$

$\Rightarrow f = \int x^2\cos(y)\,dx = \frac{1}{3}x^3\cos(y) + C(y)$

② $\frac{\partial}{\partial y} f = f_y = -\frac{1}{3}x^3\sin(y) + C'(y) = -y^2\sin(x)$

Stuck: Need $C'(y) = \ldots y \ldots$

No potential: Always check $\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}$

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Panel 17

Find potential for $\vec{F} = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$ if exists.

M

① $f_x = y^2 \Rightarrow f(x,y,z) = xy^2 + C(y,z)$

② $f_y = \cancel{2xy} + C_y = \cancel{2xy} + e^{3z}$

$\Rightarrow C_y = e^{3z}$

$\Rightarrow C(y,z) = ye^{3z} + D(z)$

$\Rightarrow f(x,y,z) = xy^2 + ye^{3z} + D(z)$

③ $f_z = \cancel{3ye^{3z}} + D'(z) = \cancel{3ye^{3z}} \Rightarrow D'(z) = 0 \Rightarrow D$ is const.

$\Rightarrow f(x,y,z) = xy^2 + ye^{3z} + \text{const}$

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Panel 18

If $F(x,y,z) = (M, N, P)$ is conservative then

$\text{curl}(F) = (0,0,0)$

Ex: Which of the following vector fields is NOT conservative:

(a) $F = \langle xy - \sin(z), \frac{1}{2}x^2 - \frac{e^y}{z}, \frac{e^y}{z^2} - x\cos z \rangle$

(b) $G = (xz, yz, xy)$

(c) $H = \langle 2xy - z^2, 2yz + x^2, y^2 - 2zx \rangle$

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Panel 19

$G = (xz, yz, xy)$ is conservative if $\operatorname{curl}(\vec{G}) = \vec{0}$

$$\operatorname{curl}(\vec{G}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} = \langle x - y, ?, ? \rangle$$

stop $\Rightarrow$ not zero

$\Rightarrow$ not conservative!

$H = \langle 2xy - z^2, 2yz + x^2, y^2 - 2zx \rangle$

$$\begin{vmatrix} i & \boxed{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - z^2 & 2yz + x^2 & y^2 - 2zx \end{vmatrix} = \langle 2y - 2y, -2z - (-2z), 2x - 2x \rangle =$$

$= \langle 0, 0, 0 \rangle \Rightarrow$ conserv.

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Panel 20

Is $F = \langle xy - \sin(z), \frac{1}{2}x^2 - \frac{e^y}{z}, \frac{e^y}{z^2} - x \cos z \rangle$ conservative?

HW

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Panel 21

Summary of Conservative Vector Field

$\vec{F} = \nabla f$, f is potential function

$\Leftrightarrow \int_C \vec{F}\, d\vec{r}$ $= f(\text{end}) - f(\text{start})$ is independent of path from A to B

$\Leftrightarrow \oint_C \vec{F}\, d\vec{r} = 0$ for all closed curves C

(If domain is simply connected) (technical)

$\operatorname{curl}(\vec{F}) = 0 \Leftrightarrow \vec{F}$ conservative

Panel 22

Note: Find curl (F) if $F(x,y) = \langle M, N \rangle$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \left\langle 0-0,\ 0-0,\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

i.e. equiv. to $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Panel 23

last Quiz

① Find a conservative vector field that has the given potential:
$f(x, y, z) = \sin(x^2 + y^2 + z^2)$

② Find ~~$div(\nabla \cdot F)$ and~~ $curl(F) = \nabla \times F$ $\quad div(F) = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$
$F(x, y, z) = \langle x^2 z, y^2 x, y + 2z \rangle$

③ Evaluate $\int_C (x - y)dx + xdy$ if C is the graph of $y^2 = x$ from (4,-2) to (4,2)
$r(t) = \langle t^2, t \rangle$, t from -2 to 2

④ Find the work done by F(x,y,z) along the curve $\langle t, t^2, t^3 \rangle$ from (0, 0, 0) to (2, 4, 8), where $F(x, y, z) = \langle y, z, x \rangle$

options < do it / f(end) - f(start) if F conservative

⑤ Check which of the following vector fields is not conservative.
$F(x, y) = \langle 3x^2 y + 2, x^3 + 4y^3 \rangle$
$F(x, y) = \langle e^x, 3 - e^x \sin(y) \rangle$
$F(x, y, z) = \langle 8xz, 1 - 6yz^2, 4x^2 - 9y^2 z^z \rangle$

⑥ Show that the line integrals are independent of the path, and find their value:
$\int_{(-1,2)}^{(3,11)} \underbrace{(y^2 + 2xy)}_{M}dx + \underbrace{(x^2 + 2xy)}_{N}dx = f(3,11) - f(-1,2)$ $\quad f = \ldots\ldots$ work!
$\int_{(1,0,2)}^{(-2,1,3)} \underbrace{(6xy^3 + 2z^2)}_{M}dx + \underbrace{(9x^2 y^2)}_{N}dy + \underbrace{(4xz + 1)}_{P}dz$

is curl $(F) = 0$?

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Panel 24

Find work done by $F = \langle x^2 + y^2, 2xy \rangle$
from $(-1,0)$ to $(10,1)$

$\int_C \vec{F}\, d\vec{r}$ < work it out / f(end) - f(start)

Must use this and F better be conservative

$F = \langle 3x^2 y + 2, x^3 + 4y^3 \rangle$ find potential:

$f_x = 3x^2 y + 2 \Rightarrow f = x^3 y + 2x + C(y)$
$f_y = \cancel{x^3} + C'(y) = \cancel{x^3} + 4y^3 \Rightarrow C'(y) = 4y^3$
$\Rightarrow C(y) = y^4 + c$

$\Rightarrow f(x,y) = x^3 y + 2x + y^4 + c$

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Panel 25

$\int_\gamma \vec{F}\, d\vec{r}$ important because it is Work

$\int_\gamma \vec{F}\, d\vec{r}$ — long way, using curve paramet., ...
— shortcut $f(B) - f(A)$ if conservative

$\oint_\gamma \vec{F}\, d\vec{r}$ — long way
— if conservative then zero
— other short cut!

means that γ is closed curve

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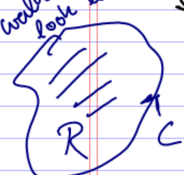
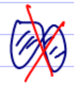
Panel 26

Green's Theorem: R a region in xy-plane with boundary curve C. C is piecewise smooth, non-intersecting, closed, and positively oriented. $\vec{F} = (M, N)$ is a smooth vector field. Then

positive = walk on C look left



$$\oint_C \vec{F}\, d\vec{r} = \oint_C M\,dx + N\,dy \overset{\text{then}}{=} \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$



Corollary: If F is conservative then $\oint_C F\, dr = 0$

already knew that any way!

either work line integr. or double integr. (if curve is closed)

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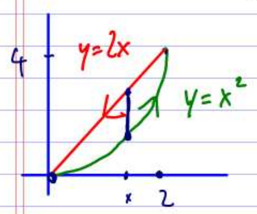
Panel 27

<u>Ex:</u> Evaluate $\oint_C \overset{M}{5xy}\,dx + \overset{N}{x^3}\,dy$, where C is as shown:



if conserv. answer is zero! Nope!

if line integral $\to \underset{\text{green curve}}{\int} + \underset{\text{red curve}}{\int}$

try Green's thm: curve closed ✓

$$\oint_\gamma F\,dr = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA = \int_0^2\int_{x^2}^{2x} 3x^2 - 5x\,dy\,dx = -\frac{28}{15}$$

27

Panel 28

<u>Ex:</u> Evaluate $\oint_C 2xy\,dx + (x^2+y^2)\,dy$, C is $4x^2+9y^2=36$

use curve the "long way": need parametrization

$r(t) = \langle \quad - \quad \rangle$

closed curve - try Green:

$$\oint_C \overset{M}{2xy}\,dx + \overset{N}{(x^2+y^2)}\,dy = \iint_{\text{ellipse}} \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA =$$

$$= \iint_{\text{ellipse}} 2x - 2x\,dA = 0$$

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Panel 29

Coming Attractions:

Stoke's + Gauss Thm

tomorrow

+ Review

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