Panel 1

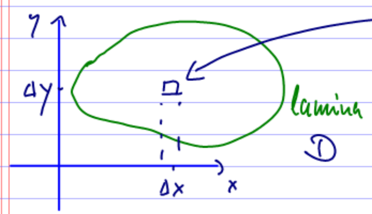
Applications of Integration

① $\iint_R f(x,y)\,dA$ = volume under $z=f(x,y)$, if $z \geq 0$

$\iint_R 1\,dA$ = area of R (Calc 2)

② **Mass of a Lamina**

Suppose we have a lamina with density function $\rho(x,y)$



if rectangle is small, then $\rho(x,y)$ is constant, approx.

$\rho(x_i,y_j)\,\Delta x\,\Delta y \sim$ mass

$$\iint_D \rho(x,y)\,dA = \lim_{(\Delta x,\Delta y)\to(0,0)} \sum\sum \rho(x_i,y_j)\,\Delta x\Delta y = \text{Mass}$$

1

Panel 2

Lamina with Density function $\rho(x,y)$

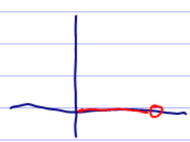
Mass: $M = \iint_D \rho(x,y)\,dA$

Moments: $M_x = \iint_D y\,\rho(x,y)\,dA$ (about x-axis)

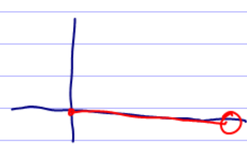
$M_y = \iint_D x\,\rho(x,y)\,dA$ (about y-axis)

Center of Gravity: $(\bar{x},\bar{y})$ where

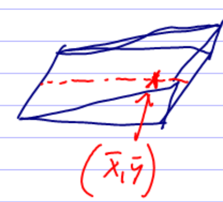
$$\bar{x} = M_y/M, \quad \bar{y} = M_x/M$$



small M_y



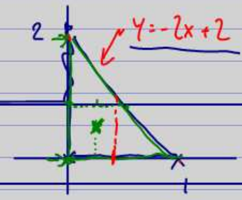
large M_y



2

Panel 3

Ex. Find center of gravity for triangular lamina with vertices (0,0), (1,), and (0,) if $\rho(x,y) = 1+3x+y$



$$M_x = \int_0^1 \int_0^{-2x+2} y(1+3x+y)\,dy\,dx = 11/6$$

$$M_y = \int_0^2 \int_0^{-\frac{1}{2}y+1} x(1+3x+y)\,dx\,dy = 1$$

$$M = \int_0^1 \int_0^{-2x+2} 1+3x+y\,dy\,dx = 8/3$$

$\bar{x} = M_y/M$, $\bar{y} = M_x/M$

$y = -2x+2$

$y-2 = -2x$

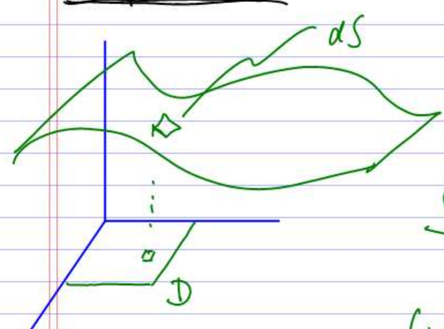
$-\frac{1}{2}y+1 = x$

$\bar{x} = 3/8 \approx 0.375$, $\bar{y} = \frac{11}{6} \cdot \frac{8}{3} = 11/16 = 0.6875$

3

Panel 4

Surface Area:



Want: Surface area of $z = f(x,y)$ over a region D is

$$S = \iint_D dS = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

$$\left(\text{i.e. } dS = \sqrt{f_x^2+f_y^2+1}\,dx\,dy\right)$$

Recall: arc length of $\vec{r}(t) = (x(t), y(t))$:

$$S = \int_a^b d\vec{r} = \int_a^b \boxed{\|r'\|\,dt} = \int_a^b \boxed{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

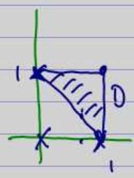
if $y = f(x)$: $S = \int_a^b \sqrt{1+(f')^2}\,dt = \int_a^b \sqrt{1+(f')^2}\,dx$

$r = \langle t, f(t) \rangle$

4

Panel 5

Ex: Surface area of $z = x^2 + 2y$ above triangle $(0,1), (1,0)$, and $(1,1)$



$$S = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA =$$

$$= \iint \sqrt{4x^2 + 4 + 1}\, dA = \int_0^1 \int_{-x+1}^1 \sqrt{4x^2+5}\, dy\, dx =$$

$$\int_0^1 \sqrt{4x^2+5} \left[\int_{-x+1}^1 1\, dy \right] dx = \int_0^1 \sqrt{4x^2+5}\, (1 - (-x+1))\, dx =$$

$$= \int_0^1 x \sqrt{4x^2+5}\, dx = \frac{2}{3} \cdot \frac{1}{8} (4x^2+5)^{3/2} \Big|_0^1$$

$(4x^2+5)^{1/2}$

done

5

Panel 6

Ex: Surface area of $z = x^2 + 2y$ above square $[0,1] \times [0,1]$?

$$\int_0^1 \int_0^1 \sqrt{4x^2+5}\, dx\, dy = \int_0^1 \int_0^1 \sqrt{4x^2+5}\, dy\, dx = \int_0^1 \sqrt{4x^2+5}\, dx =$$

$$= \int_0^1 \sqrt{5\left(\tfrac{4}{5}x^2 + 1\right)}\, dx = \sqrt{5} \int_0^1 \sqrt{\tfrac{4}{5}x^2 + 1}\, dx =$$

$$= \sqrt{5} \int_0^1 \sqrt{\left(\sqrt{\tfrac{4}{5}}\, x\right)^2 + 1}\, dx = \sqrt{5}\, \frac{1}{\sqrt{\frac{4}{5}}} \int \sqrt{u^2+1}\, du =$$

$\sqrt{4/5}$

$u = \sqrt{\frac{4}{5}}\, x$

$du = \sqrt{\frac{4}{5}}\, dx$

$$= \frac{5}{2} \int_0 \sqrt{u^2+1}\, du = \frac{5}{2} \int \sec^3(t)\, dt = 2.5$$

~~$u = \sin(t)$~~

$u = \tan(t)$

$du = \sec^2(t)\, dt$

$u = \sec(t)$, $u' = \sec(t)\tan(t)$

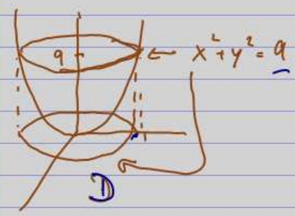
$v' = \sec^2(t)$, $v = \tan(t)$

HW

6

Panel 7

Ex: Area of paraboloid $z = x^2 + y^2$ under $z = 9$



$$S = \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA \underset{x = r\cos(\theta)}{\overset{\text{polar}}{=}}$$

$$= \int_0^{2\pi} \int_0^3 \sqrt{4r^2+1}\; r\,dr\,d\theta = \text{easy!}$$

7

Panel 8

Reading assignment: (no class on Thursday)

Moments of inertia, radius of gyration → 16.5

Probability density function → 16.5

Read on Thursday

also Ex 3, chapter 16.7

Tripple integrals 16.7

Thursday's work

8

Panel 9

Birds-Eye View so far

$f: \mathbb{R} \to \mathbb{R}$ Calc 1/2

$f: \mathbb{R} \to \mathbb{R}^2$ space curves $r(t) = \langle \cos(t), \sin(t) \rangle$

$f: \mathbb{R}^2 \to \mathbb{R}$ surface $f(x,y) = x^2 + y^2$

($\mathbb{R}^3 \to \mathbb{R}$ similar)

Next: $f: \mathbb{R}^2 \to \mathbb{R}^2$ called vector field.

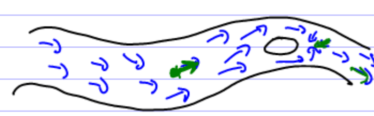
$\nexists$ "function" does some domain, $\mathbb{R}$ as range

9

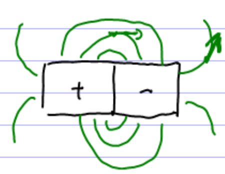
Panel 10

Vector Fields: If for each point P in a region R there is a unique vector having initial point P, then the totality of such vectors is called a vector field.

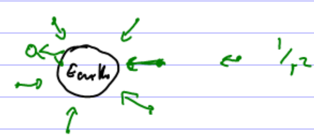
Ex: Flow of water



Ex: Magnetic Field



Ex: Gravity



10

Panel 11

Mathematically, a vector field is given as:

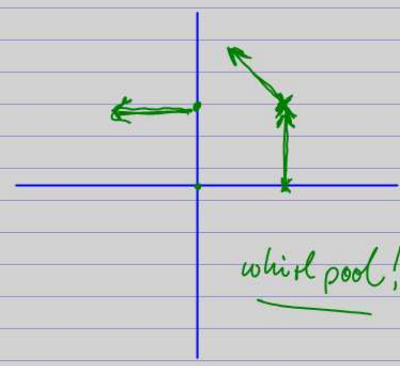
$$F(x,y) = \langle M(x,y), N(x,y)\rangle = M(x,y)\vec{i} + N(x,y)\vec{j}, \text{ book}$$

$$F(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle = M\vec{i} + N\vec{j} + P\vec{k}$$

$\mathbb{R}^3$ $\mathbb{R}^3$ $M(x,y) = -y$, $N(x,y) = x$

Ex: Describe $F(x,y) = \langle -y, x\rangle = -y\vec{i} + x\vec{j}$

(x,y)	$F(x,y)$
0,0	$\langle 0,0\rangle$
1,0	$\langle 0,1\rangle$
0,1	$\langle -1,0\rangle$
1,1	$\langle -1,1\rangle$



whirlpool!

11

Panel 12

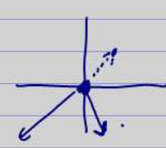
Def: If $r(x,y,z) = \langle x,y,z\rangle$ then $F(x,y,z) = \frac{c}{\|r\|^2}\vec{u}$

where $u = r/\|r\|$ is called inverse square field.

Ex: Describe inverse square field for $c = -1$.

$$F(x,y,z) = -\frac{1}{\|r\|^2}\vec{u} = -\frac{1}{\|r\|^2}\frac{r}{\|r\|} = -\frac{1}{\|r\|^3}\langle x,y,z\rangle =$$

$$= -\underbrace{\frac{1}{(x^2+y^2+z^2)^{3/2}}}_{\text{scalar}}\langle x,y,z\rangle$$



at any point it

points to origin

longer if closer

12

Panel 13

Maple offers "fieldplot" and "fieldplot3d"

```
> with(plots);
> fieldplot([-y, x], x = -2..2, y = -2..2);
> fieldplot3d([-x/(x^2+y^2+z^2)^(3/2), -y/(x^2+y^2+z^2)^(3/2), -z/(x^2+y^2+z^2)^(3/2)], x = -2..2, y = -2..2, z = -2..2);
>
```

difficult to visualize

very cool looking

13

Panel 14

Def: Suppose $F(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$

Then $$\text{curl}(F) = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

$$\text{div}(F) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Hints consider operator $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$$\text{curl}(F) = \nabla \times F = \begin{vmatrix} i & \boxed{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \ldots \right\rangle$$

$$\text{div}(F) = \nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle M, N, P \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

14

Panel 15

Ex: Let $F(x,y,z) = \langle xy,\ yz,\ xz \rangle$. Then

curl (F):

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} = \langle 0-y,\ -z-0,\ 0-x \rangle$$

$$= \langle -y,\ -z,\ -x \rangle$$

div (F): $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle xy, yz, xz \rangle = y + z + x$

15

Panel 16

Ex! $F(x,y,z) = \langle xy^2z^4,\ 2x^2y + z,\ y^3z^2 \rangle$

Find curl (F) and div (F)

$\mathrm{div}(F) = y^2z^4 + 2x^2 + 2y^3z$

$$\mathrm{curl}(F) = \begin{vmatrix} i & j & \boxed{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^4 & 2x^2y+z & y^3z^2 \end{vmatrix} =$$

$$= \langle 3y^2z^2 - 1,\ +4xy^2z^3,\ 4xy - 2xyz^4 \rangle$$

16

Panel 17

$f(x,y) = x^2 \cdot y^2 \quad : \mathbb{R}^2 \to \mathbb{R}$ function

$\vec{F}(x,y) = \langle x^2, y^2, 0 \rangle : \mathbb{R}^2 \to \mathbb{R}^2$ vector field

$\cancel{\text{curl}}(f)$ — ∇f

$\text{curl}(F)$ ✓

$\text{div}(\text{curl}(F))$ ✓ — $\cancel{\nabla F}$

$\text{curl}(\cancel{\text{div}(F)})$ — $\nabla \cdot F = \text{div}(F)$

$\text{curl}(\text{curl}(F))$ ✓ — $\cancel{\nabla \cdot f}$

$\text{div}(\cancel{\text{div}}(F))$ — $\nabla \times F = \text{curl}(F)$

17

$\cancel{\nabla \times f}$

Panel 18

Def: A vector field $\vec{F}$ is conservative if there is a function $f : \mathbb{R}^3 \to \mathbb{R}$ s.t. $\nabla f = \vec{F}$.
The function f is the potential function for $\vec{F}$

Ex: Find vector field with potential

$f(x,y,z) = x^2 - 3y^2 + 4z^2$

$\Rightarrow \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle 2x, -6y, 8z \rangle$

Thus: $F(x,y,z) = \langle 2x, -6y, 8z \rangle$ is conservative

with potential function $x^2 - 3y^2 + 4z^2$

"potential is antideriv of vector field"

18

Panel 19

Which of the following vector field(s) has as potential function $f(x,y,z) = x^2y^2z^2 + xy + zy$

(a) $\vec{F} = \langle 2x, y, \underline{z} \rangle$

(b) $\vec{F} = \langle 2xy^2z^2 + x + y \rangle$

(c) $\vec{F} = \langle 2xy^2z^2, y, \underline{z} \rangle$

(d) $\vec{F} = \langle 2xy^2z^2, x, \underline{y} \rangle$ ✓

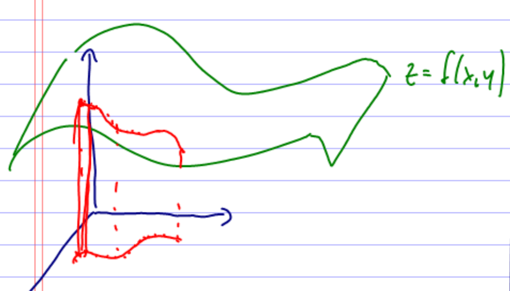
19

Panel 20

Line Integrals

Suppose $r(t) = \langle g(t), h(t) \rangle$ describes a curve C in $\mathbb{R}^2$ and $z = f(x,y)$ is a function defined on C. Then we define the line integral of f along C as:

$$\int_C f(x,y)\, \underline{d\vec{r}} = \int f(x,y) \sqrt{(g')^2 + (h')^2}\, dt =$$

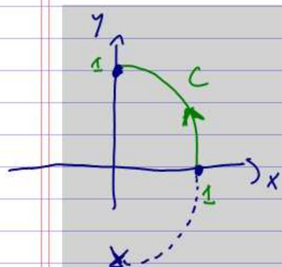


$$= \int f(g(t), h(t)) \sqrt{(g')^2 + (h')^2}\, dt$$

20

Panel 21

Ex: Find $\int_C xy^2\, d\vec{r}$, where $\vec{r}$ describes curve C as first quarter circle, radius 1.



$$\int_C xy\, d\vec{r} = \int_0^{\pi/2} \underbrace{\cos(t)}_{x}\cdot\underbrace{\sin(t)}_{y}\sqrt{(-\sin)^2+\cos^2}\,dt$$

$$= \int_0^{\pi/2} \cos(t)\sin(t)\,dt$$

$r(t) = \langle \sin(t), \cos(t)\rangle$, $t = \frac{\pi}{2}$ to 0

$t=0$: $\langle 0,1\rangle$

$t=\pi$: $\langle 0,-1\rangle$

$t=\frac{\pi}{2}$: $\langle 1,0\rangle$

or $\boxed{r(t) = \langle \cos(t), \sin(t)\rangle, \quad t = 0 \text{ to } \pi/2}$

21

Panel 22

Final example: Let $r(t) = \langle \underset{x}{t}, \underset{y}{t^2}\rangle$, $t=-1,1$,

$f(x,y) = x^2+y^2$, $F(x,y) = \langle x^2, y^2\rangle$

Find, if you can:

$\cancel{\int_{-1}^{1} f\,dx}$ $\qquad \iint f\,dA$ ✓

$\Rightarrow \int_C f\,d\vec{r} = \int_{-1}^{1}(t^2+t^4)\sqrt{1+4t^2}\,dt$ $\qquad \cancel{\iint F\,dA}$

$\cancel{\int_{-1}^{1} F\,d\vec{r}}$ $\qquad$ Maple!

$\cancel{\int_{-1}^{1} F\,dx}$

22