Panel 1

Finding (relative) Max/Min for $z = f(x,y)$

① $\nabla f = \langle f_x, f_y \rangle$

② $\nabla f = \vec{0} \iff \begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}$

③ $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ ~ Hessian matrix

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

a) $D > 0$, $f_{xx} > 0$ $\Rightarrow f$ min

b) $D > 0$, $f_{xx} < 0$ $\Rightarrow f$ max

c) $D < 0$ $\Rightarrow f$ saddle

d) $D = 0$ $\Rightarrow$ NO CLUE

1

Panel 2

Suppose $f(x,y) = x^2 + 2y^2 + 4xy$. Find and classify all relative extrema, if any.

① $f_x = 2x + 4y$ ② $2x + 4y = 0$ $2x - 4x = 0 \Rightarrow x = 0$

$f_y = 4y + 4x$ $4y + 4x = 0 \Rightarrow y = -x \Rightarrow y = 0$

③ $H = \begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix}$, $D = 8 - 16 = -8$

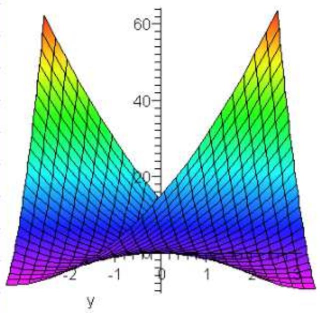
$\rightarrow (0,0)$

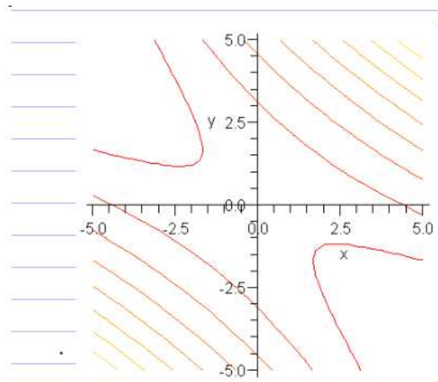
critical point is saddle point!

2

Panel 3

Suppose $f(x,y) = x^2 + 2y^2 + 4xy$. Find and classify all relative extrema, if any.



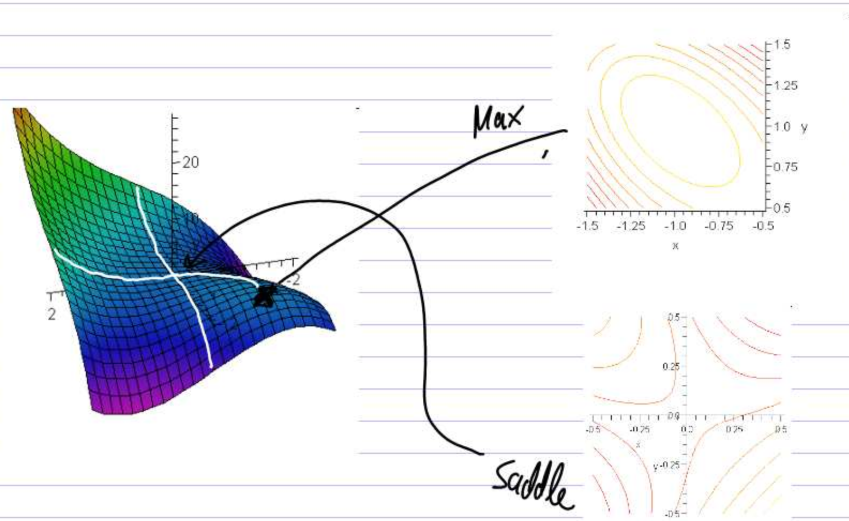


3

Panel 4

Ex: Find and classify the critical points for

$f(x,y) = x^3 - y^3 - 3xy$



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Panel 5

Ex: Find and classify the critical points for

$f(x,y) = x^3 - y^3 - 3xy$

$f_x = 3x^2 - 3y$ $\qquad$ $f_x = 0: \; 3x^2 - 3y = 0 \iff x^2 = y$

$f_y = -3y^2 - 3x$ $\qquad$ $f_y = 0: \; -3y^2 - 3x = 0$

critical: $(0,0)$ and $(-1,1)$ $\qquad$ $-3x^4 - 3x = -3x(x^3+1) = 0$

$H = \begin{pmatrix} 6x & -3 \\ -3 & -6y \end{pmatrix}$ $\qquad$ at $(0,0)$: $D = -9 \Rightarrow$ saddle

$D = -36xy - 9$ $\qquad$ at $(-1,1)$: $\left.\begin{matrix} D = 27 \\ f_{xx} < 0 \end{matrix}\right\} \Rightarrow$ max

Answer: $(0,0)$ is saddle, $(-1,1)$ is max!

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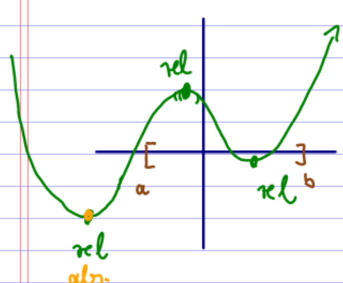
Panel 6

Absolute Max / Min

Differences between relative and absolute extrema?



3 relative extrema

1 absolute extrema

if f is restricted to $[a,b]$ then
$\Rightarrow$ 1 abs. max, 1 abs. min.

Thm $f(x,y)$ is continuous on a closed, bounded set D in $\mathbb{R}^2$. Then it has an abs max and min. They can occur either inside D or on the boundary!

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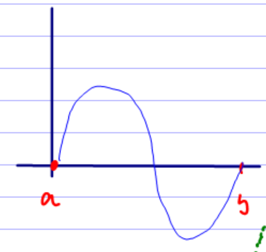
Panel 7

f 1 abs. max

1 abs. min

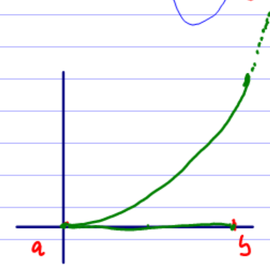
2 rel. extrema



f 1 abs. max

1 abs. min

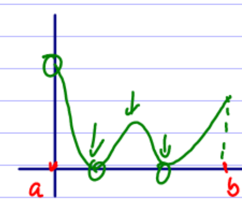
no relative extrem



f 1 abs. max

2 abs. min

3 rel. extrema



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Panel 8

To find abs max / min:

If $f(x,y)$ is cont. on closed bdd. set D then

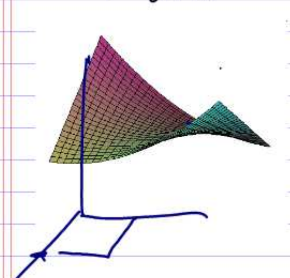
a) find critical points inside D

⟹ b) find critical points on the boundary of D

c) find smallest / largest value of (a) and (b)

Ex: Find abs max / min of $f(x,y) = x^2 - 2xy + 2y$ on rectangle

$D = \{(x,y) : 0 \le x \le 3,\ 0 \le y \le 2\}$



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Panel 9

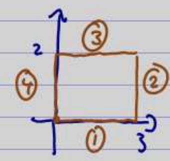
Ex: Find abs. extrema for $f(x,y) = x^2 - 2xy + 2y$ on $\overset{x}{[0,3]} \times \overset{y}{[0,2]}$

There is at least one abs. max / min.

① $f_x = 2x - 2y$ $2x - 2y = 0$ $y = 1$ critical $(1,1)$

$f_y = -2x + 2$ $-2x + 2 = 0$ $\Rightarrow x = 1$

② boundary:



① $y = 0$, $0 \le x \le 3$: $f(x,y) = f(x) = x^2$

Calc I problem $\Rightarrow x = 0$, $y = 0$ critical

$(0,0)$, $(3,0)$ endpoints

② $x = 3$, $0 \le y \le 2$: $f(x,y) = f(y) = 9 - 4y$

$(3,0)$, $(3,2)$ endpoints

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Panel 10

③ $y = 2$, $0 \le x \le 3$: $f(x,y) = f(x) = x^2 - 4x + 4 = (x-2)^2$

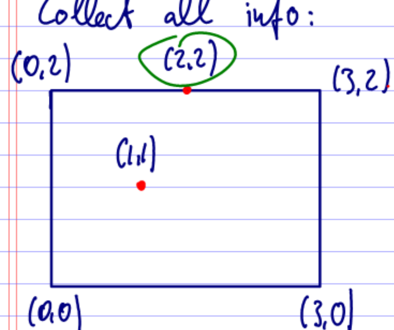
critical: $(2,2)$

end: $(0,2)$, $(3,2)$

④ $0 \le y \le 2$, $x = 0$: $f(x,y) = f(y) = 2y$

end: $(0,0)$, $(0,2)$

Collect all info:



$f(x,y) = x^2 - 2xy + 2y$

(x,y)	$f(x,y)$
$(0,0)$	0
$(3,0)$	9
$(3,2)$	$9 - 12 + 4 = 1$
$(0,2)$	4
$(2,2)$	$4 - 8 + 4 = 0$
$(1,1)$	$1 - 2 + 2 = 1$

abs. max

abs. min

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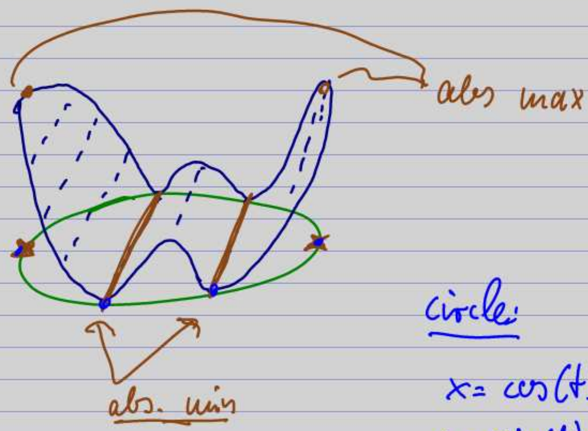
Panel 11

Ex: Find abs. extrema of $f(x,y) = \underline{(x^2 - \tfrac{1}{4})^2}$ in unit disk.



circle:

$x = \cos(t)$

$y = \sin(t)$

on bdd: $f(x,y) = f(t) = \left(\cos^2(t) - \frac{1}{4}\right)^2, \ t \in [0, 2\pi]$

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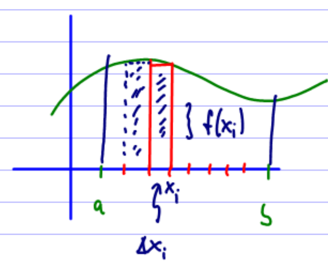
Panel 12

Done with 15

Integration:

In $\mathbb{R}$:



$f(x_i)\Delta x_i$ is area of rectangle

$$\int_a^b f(x)\,dx = \lim_{\|\Delta x\| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

(Riemann) integral is defined as limit of Riemann sums

$\sim$ gives area under curve

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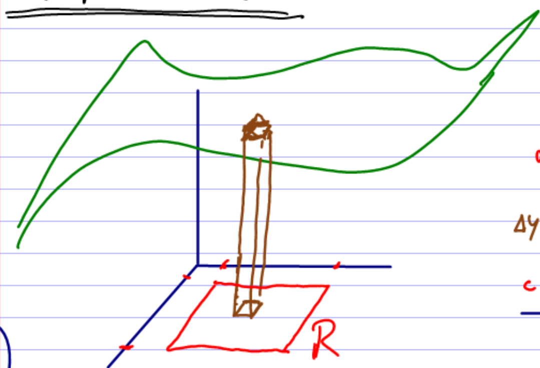
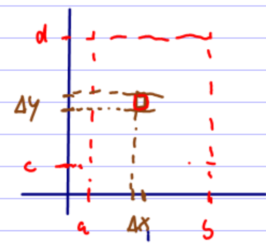
Panel 13

Integration in $\mathbb{R}^2$



$f(x_i, y_j)\Delta x_i \Delta y_j$ gives volume



Def

$$\iint_R f(x,y)\, dA = \lim_{(\Delta x, \Delta y)\to(0,0)} \sum_{i,j=1}^{n} f(x_i, y_j)\, \Delta x_i \Delta y_j$$

A for area

is geometrically volume under surface!

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Panel 14

Fubini's Theorem (how to integrate in $\mathbb{R}^2$)

If f is continuous on rectangle $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

then $$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Ex: Find $\iint_R (x - 3y^2)\, dA$, $R = [0,2] \times [1,2]$

i.e. "volume" under surface $x - 3y^2$ as $x \in [0,2]$, $y \in [1,2]$

$$\iint_R (x-3y^2)\, dA = \int_1^2 \int_0^2 x - 3y^2\, dx\, dy = \int_1^2 \left(\frac{1}{2}x^2 - 3y^2 x \Big|_{x=0}^{x=2} \right) dy$$

$$= \int_1^2 2 - 6y^2\, dy = 2y - 2y^3 \Big|_1^2 =$$

$$= (4-16) - (2-2) = -12$$

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Panel 15

$$\iint_R x-3y^2\,dA = \int_0^2\int_1^2 x-3y^2\,dy\,dx =$$

$$= \int_0^2 xy - y^3 \Big|_{1=y}^{2=y} dx =$$

$$= \int_0^2 (2x-8)-(x-1)\,dx =$$

$$= \int_0^2 x-7\,dx = \frac{1}{2}x^2-7x\Big|_0^2 = 2-14-0$$

$$= \underline{\underline{-12}}$$

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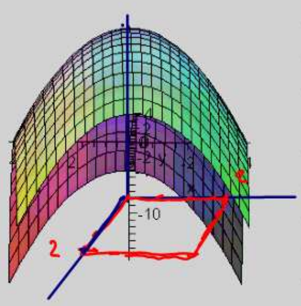
Panel 16

Ex: Find volume of solid bounded by $x^2+y^2+z=16$, the planes $x=2$ and $y=2$, and the coordinate planes



$$z = 16-x^2-y^2$$

$$V = \iint_R f(x,y)\,dA =$$

$$= \int_0^2\int_0^2 16-x^2-y^2\,dx\,dy =$$

$$= \int_0^2 16x - \frac{1}{3}x^3 - y^2x\Big|_{0=x}^{2=x} dy = \int_0^2 32-\frac{8}{3}-2y^2\,dy =$$

#

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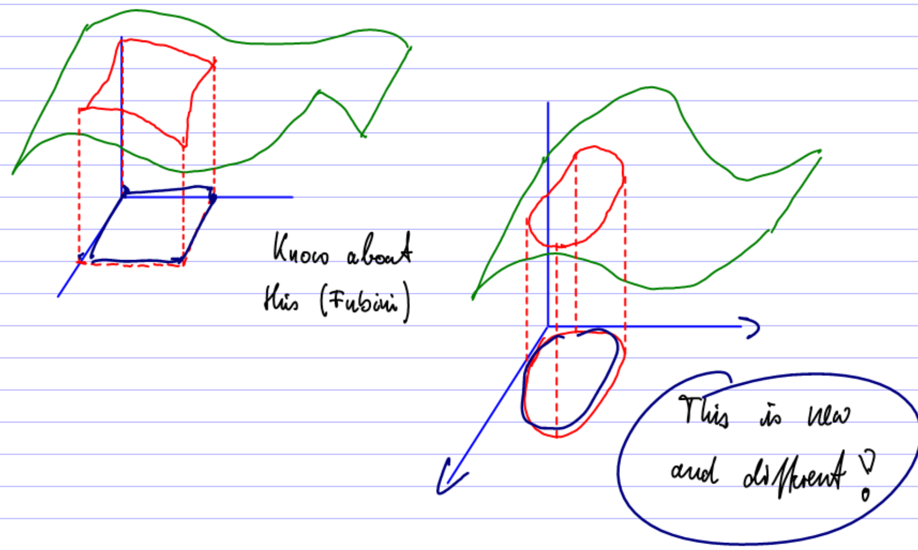
Panel 17

In $\mathbb{R}$ all we ever did was integrate over intervals $[a,b]$.

How about integration in $\mathbb{R}^2$?



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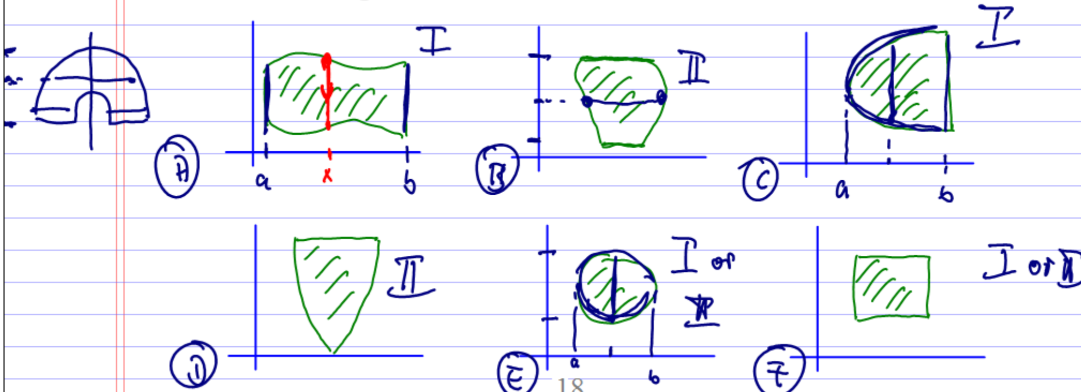
Panel 18

<u>Thm:</u> A <u>type 1 region</u> is $D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

$\Rightarrow \iint_D f(x,y)\,dA =$

A <u>type 2 region</u> is $D = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

$\Rightarrow \iint_D f(x,y)\,dA =$



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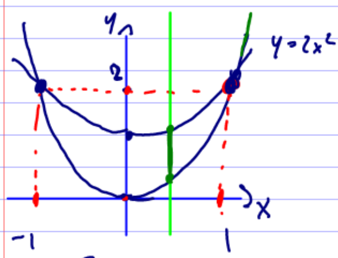
Panel 19

Ex: Evaluate $\iint_D (x+2y)\,dA$ where D is region bounded by $y=2x^2$ and $y=1+x^2$



$$2x^2 = 1+x^2$$
$$x^2 = 1 \qquad x=\pm 1,\ y=2$$

$$\cancel{\int_0^2\int dx\,dy}$$

$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} x+2y\,dy\,dx = \int_{-1}^{1} xy+y^2\Big|_{2x^2}^{1+x^2} dx = \int_{-1}^{1}\left(x(1+x^2)+(1+x^2)^2\right)-\left(x(2x^2)-(2x^2)^2\right)dx = \int_{-1}^{1} \dots x^4 \dots\, dx = \#$$

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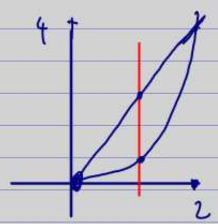
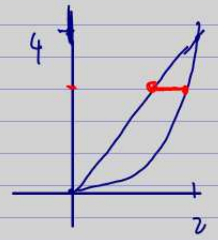
Panel 20

Ex: Volume under $z=x^2+y^2$ above $y=2x$ and $y=x^2$.



$$\int_0^2\int_{x^2}^{2x} x^2+y^2\,dy\,dx = \text{easy}$$

HW



$$\int_0^4\int_{\cancel{2x}\ y/2}^{\cancel{x^2}\ \sqrt{y}} x^2+y^2\,dx\,dy = \text{easy}$$

$$y=2x \iff x=\tfrac{1}{2}y$$
$$y=x^2 \iff x=\sqrt{y}$$

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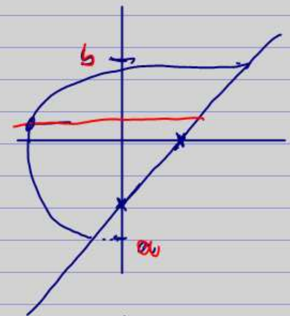
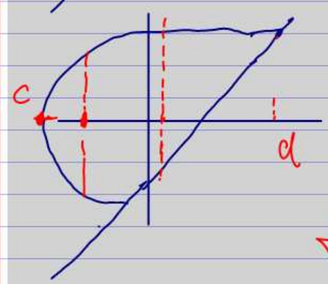
Panel 21

Ex: Find $\iint_D xy\,dA$ where D is bounded by $y=x-1$ and $y^2=2x+6$. Should you $\iint xy\,dxdy$ or $\iint xy\,dydx$?



$$\int_a^b\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dxdy = \text{HW}$$

good!



$$\int_c^d\int xy\,dydx$$

two integrals necessary

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Panel 22

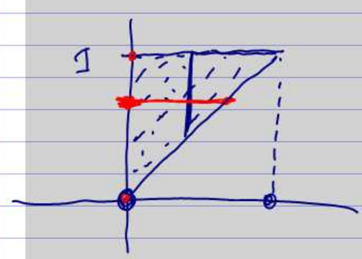
Ex: Find $\int_0^1\int_x^1 \sin(y^2)\,dy\,dx =$

$y=x$ $y=1$

would need antideriv. of $\sin(y^2)$ w.respt. to y

$\Rightarrow$ impossible!

BUT: $= \int_0^1\int_0^y \sin(y^2)\,dx\,dy$ is easier (hint)



$y=x$

$y=1$

$$\int_0^1 x\sin(y^2)\Big|_0^y dy =$$

$$\int_0^1 y\sin(y^2)-0\,dy =$$

$$\int_0^1 y\sin(y^2)\,dy = -\frac{1}{2}\cos(y^2)\Big|_0^1$$

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