Panel 1

Last time:

Partial derivative, tangent plane, review of limits

$f(x,y) = x^3 + 3x^2y^3 - y^4$

a) $f_x(x,y) = 3x^2 + 6xy^3 - 4y^3$

b) $f_x(x,y) = 3x^2 + 18xy^2 - 4y^3$

c) $f_x(x,y) = 3x^2 + 6xy^3$

d) $f_x(x,y) = 3x^2 + 6x$

e) $f_x(x,y) = 3x^2 + 9x^2y^2$

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Panel 2

Quiz 5 - part 1

If it exists, find $\lim_{(x,y)\to(0,0)} \frac{2y^2+3}{4x^2+5y^2+6}$

If it exists, find $\lim_{(x,y)\to(0,0)} \frac{2y^2}{3x^2+4y^2}$

$x=0$

$y=0$

If it exists, find $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

$y=x^2$

$x=y^2$ $\Big\}$ $\frac{y^4}{y^4+y^4} \to \frac{1}{2}$

$x=0 \to 0$

$y=0 \to 0$

$x=y \quad \lim_{y\to 0} \frac{y^3}{y^2+y^4} = \lim_{y\to 0} \frac{3y^2}{2y+4y^3} = \lim \frac{6y}{2+12y^2} = \underline{\underline{0}}$

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Panel 3

Quiz 5 - Part 2

$\frac{d}{dx}\left(x \cdot \sin(5x)\right) = \sin(5x) + x\cos(5x)\cdot 5$

Suppose $f(x,y) = x\sin(xy)$

Find $\frac{\partial}{\partial x} f(x,y)$ $= 1 \cdot \sin(xy) + x \cdot \cos(xy)\cdot y$

$f_y = x\cos(xy)\cdot x = x^2\cos(xy)$

$f_{yy} = x^2(-\sin(xy))\cdot x = -x^3\sin(xy)$

Find f_{yy}

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Panel 4

Equation of tangent plane to $z = f(x,y)$ at (x_0, y_0) is:

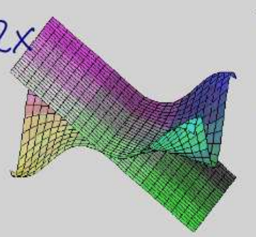
$$z = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + z_0$$

Ex: $f(x,y) = 2x\cos(xy)$. Find tangent plane to f at $P(0,0,0)$.

$f_x = 2\cdot\cos(xy) + 2x\cdot(-\sin(xy))\cdot y \Big|_{(0,0)} = 2$

$f_y = 2x(-\sin(xy))\cdot x \Big|_{(0,0)} = 0$

$z = 2(x-0) + 0 + 0 = 2x$



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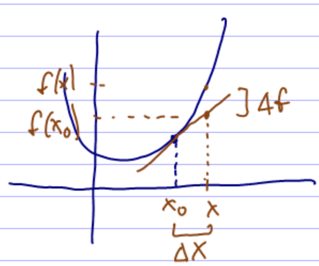
Panel 5

Want to define differentiability:

1 variable: $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

2 variable: $f'(x_0, y_0) = \lim_{(x,y) \to (x_0, y_0)} \frac{f(x,y) - f(x_0, y_0)}{(x,y) - (x_0, y_0)}$ ⇐ vector!

Deriv in $\mathbb{R}$ differently:



$$\frac{f(x) - f(x_0)}{x - x_0} \approx f'(x_0)$$

$$f(x) - f(x_0) \approx f'(x_0)(x - x_0)$$

$$\Delta f = f'(x_0)\,\Delta x + \text{Error}$$

i.e. f diff'ble if it can be approx by tangent line

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Panel 6

Can use this to define concept of differentiable:

Def: $f(x,y)$ is differentiable at (a,b) if Δf can be expressed:

$$\Delta f \approx \underbrace{f_x \,\Delta x + f_y \,\Delta y}_{\text{tangent plane}} + \text{error}$$

where $\text{Error} \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

In words: $f(x,y)$ is diff'ble if it has a tangent plane

Thm: If $f(x,y)$ has f_x and f_y, and f_x, f_y are cont., then f is diff'ble

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Panel 7

Def: If $f(x,y)$ is differentiable then the total differential is:

$$df = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Ex: Find the total differential for $f(x,y) = x^2 + 3xy - y^2$

$$f_x = 2x + 3y$$

$$f_y = 3x - 2y$$

$$df = f_x\,dx + f_y\,dy =$$

$$= (2x+3y)\,dx + (3x-2y)\,dy$$

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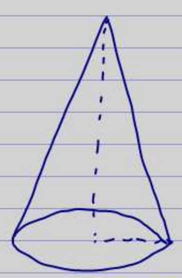
Panel 8

Ex: The base radius and height of a right circular cone are 10 cm and 25 cm, resp., with possible error ± 0.1 cm. Estimate error in volume of the cone.



$$V = \frac{1}{3}\pi r^2 h =$$

height $= 25 \pm 0.1$

radius $= 10 \pm 0.1$

Want: $dV = V_r\,dr + V_h\,dh$

$$V_r = \frac{2}{3}\pi r h = \frac{2}{3}\pi \cdot 10 \cdot 25 = \frac{500}{3}\pi$$

$$V_h = \frac{1}{3}\pi r^2 = \frac{1}{3}\pi\, 10^2 = \frac{100}{3}\pi$$

$\rightarrow$ error: $dV = V_r\,dr + V_h\,dh = \frac{500}{3}\pi \cdot 0.1 + \frac{100}{3}\pi \cdot 0.1 = \underline{\underline{20\pi}}$

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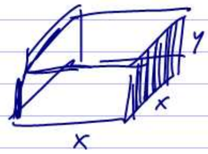
Panel 9

A box has a square base with side length x and a height of length y. When measured, $x = 10$ cm and $y = 20$ cm, both with an error of 0.1 cm. The total surface area of that box is [illegible]. What is the error in the computed surface area, given that both x and y are measured with error 0.1 cm.



$$S(x,y) = 4xy + 2x^2$$

$$\Rightarrow S(10,20) = 4\cdot 10\cdot 20 + 2\cdot 10^2 = 800 + 200 = 1000$$

$$dS = S_x\, dx + S_y\, dy =$$

$$= (4y + 4x)\, dx + (4x)\, dy =$$

$$= (4\cdot 20 + 4\cdot 10)\cdot 0.1 + 4\cdot 10\cdot 0.1 =$$

$$= 12 + 4 = 16$$

Surface area is $\underline{\underline{1000 \pm 16 \text{ cm}^2}}$

$f(x,y,z,w)$

$df = f_x\, dx + f_y\, dy + f_z\, dz + f_w\, dw$

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Panel 10

The Chain Rule

$f: \mathbb{R} \to \mathbb{R},\ g: \mathbb{R} \to \mathbb{R} \Rightarrow \underline{\frac{d}{dx} f(g(x))} = \underbrace{f'(g(x))\cdot g'(x)}$

Chain Rule in $\mathbb{R}^2$:

$z = f(x,y),\ x = g(t),\ y = h(t)$

$$\Rightarrow \frac{\partial f}{\partial t} = \underbrace{\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}} + \underbrace{\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}} \Leftarrow$$

more applicable

$z = f(x,y),\ x = g(s,t),\ y = h(s,t)$

$$\Rightarrow \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

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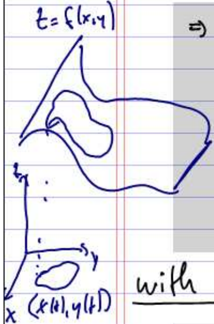
Panel 11

Ex: $f(x,y) = x^2 y + 3xy^4$, $x = \sin(2t)$, $y = \cos(t)$. Find $\frac{\partial f}{\partial t}$ at $t=0$

without chain rules: $x = \sin(2t)$, $y = \cos(t)$, $z = x^2 y + 3xy^4$



$\Rightarrow f(t) = \sin^2(2t)\cdot\cos(t) + 3\sin(2t)\cdot\cos^4(t)$

$$f'(t) = 2\sin(2t)\cdot\cos(2t)\cdot 2\cdot\cos(t) + \sin^2(2t)\left(-\sin(t)\right) + 6\cos(2t)\cos^4(t) + 3\sin(2t)\cdot 4\cos^3(t)(-\sin(t))\Big|_{t=0} = 6$$

with chain rules: $x = \sin(2t)$, $x(0)=0$; $y(t) = \cos(t)$, $y(0) = 1$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 3\cdot 2 + 0\cdot 0 = \underline{6}$$

$\frac{\partial f}{\partial x} = 2xy + 3y^4\Big|_{(0,1)} = 3$ $\qquad \frac{\partial x}{\partial t} = 2\cos(2t)\Big|_{t=0} = 2$

$\frac{\partial f}{\partial y} = x^2 + 12xy^3\Big|_{(0,1)} = 0$ $\qquad \frac{\partial y}{\partial t} = -\sin(t)\Big|_{t=0} = 0$

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Panel 12

Chain Rule is useful for Implicit Differentiation

Suppose $F(x,y) = 0$ is a function defining x and y implicitly, or more precisely, defines $y = y(x)$ implicitly.

$x = x$, $y = y(x)$

Chain rule: $F(x,y) = 0 \quad \Big| \frac{\partial}{\partial x}$ $\qquad x, y$ are functions of x

$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0$$

$$\frac{\partial F}{\partial x}\cdot 1 = -\frac{\partial F}{\partial y}\frac{\partial y}{\partial x} \quad \Rightarrow \quad \frac{\partial y}{\partial x} = -F_x / F_y$$

Similarly: $F(x,y,z) = 0$ defines z as functions of (x,y)

$$\frac{\partial z}{\partial x} = -F_x / F_z, \quad \frac{\partial z}{\partial y} = -F_y / F_z$$

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Panel 13

<u>Ex:</u> Find y' if $x^3+y^3=6xy \iff F(x,y)=0 \iff \overbrace{x^3+y^3-6xy}^{F(x,y)}=0$

$$\frac{\partial y}{\partial x} = -F_x/F_y = -\frac{3x^2-6y}{3y^2-6x} = y'$$

<u>Ex:</u> Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1 \iff F(x,y,z)=x^3+y^3+z^3+6xyz-1=0$

This defines z as a function of (x,y), e.g. $z=1$ if $x=y=0$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$= \frac{\partial F}{\partial x} + 0 + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -F_x/F_z$$

$$\frac{\partial z}{\partial x} = -F_x/F_z = -\frac{3x^2+6yz}{3z^2+6xy}$$

$$\frac{\partial z}{\partial y} = -F_y/F_z = -\frac{3y^2+6xz}{3z^2+6xy}$$

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Panel 14

<u>Directional Derivatives:</u>

f_x is deriv. in x direction: $f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$

f_y is deriv. in y-direction:

What about derivative in direction of $\vec{u}=\langle a,b\rangle$

<u>Def:</u> Directional derivative of $f(x,y)$ in direction of unit vector $\vec{u}=\langle a,b\rangle$ is:

$$D_{\vec{u}}(f) = \lim_{h\to 0}\frac{f(x+ha,y+hb)-f(x,y)}{h}$$

if $\vec{u}=\langle 1,0\rangle$: $D_{\vec{u}}(f)=f_x$

$\vec{u}=\langle 0,1\rangle$: $D_{\vec{u}}(f)=f_y$

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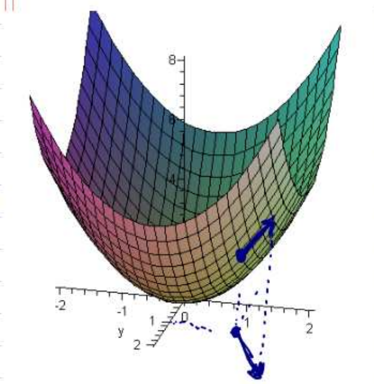
Panel 15

Ex: $f(x,y) = x^2 + y^2$

Find directional derivative in direction of $\vec{u} = \frac{1}{\sqrt{2}}\langle 1,1\rangle$ at $(1,1)$



1. Is $\vec{u}$ unit vector? YES

2. $$D_{\vec{u}}(f) = \lim_{h\to 0} \frac{f(x+\frac{1}{\sqrt{2}}h, y+\frac{1}{\sqrt{2}}h) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{(x+\frac{1}{\sqrt{2}}h)^2 + (y+\frac{1}{\sqrt{2}}h)^2 - (x^2+y^2)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + \sqrt{2}hx + \frac{1}{2}h^2 + \cancel{y^2} + \sqrt{2}hy + \frac{1}{2}h^2 - \cancel{x^2} - \cancel{y^2}}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{h}(\sqrt{2}x + \frac{1}{2}h + \sqrt{2}y + \frac{1}{2}h)}{\cancel{h}} = \sqrt{2}x + \sqrt{2}y \Big|_{(1,1)} = \underline{\underline{2\sqrt{2}}}$$

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Panel 16

Thm: Suppose $\vec{u} = \langle a,b\rangle$ is a unit vector. Then

$$D_{\vec{u}} f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\, b = \langle f_x, f_y\rangle \cdot \langle a,b\rangle$$

Ex: $f(x,y) = x^2 + y^2$, $\vec{u} = \frac{1}{\sqrt{2}}\langle 1,1\rangle$. Find $D_{\vec{u}} f$ at $(1,1)$

1. Is $\vec{u}$ unit vector ✓

2. $f_x = 2x$, $f_y = 2y$ $\Rightarrow \langle f_x, f_y\rangle = \langle 2x, 2y\rangle$

$$\Rightarrow D_{\vec{u}}(f) = \langle f_x, f_y\rangle \cdot \langle a,b\rangle = \langle 2x, 2y\rangle \cdot \frac{1}{\sqrt{2}}\langle 1,1\rangle =$$

$$= \langle 2x, 2y\rangle \cdot \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle = \sqrt{2}x + \sqrt{2}y$$

$$\Rightarrow D_{\vec{u}}(f)\Big|_{(1,1)} = 2\sqrt{2}$$

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Panel 17

Def: If $f(x,y)$ is a function with partial deriv., then the vector

$$\nabla f = \langle f_x, f_y \rangle$$

grad

is called the gradient

Note: $f(x,y)$ is scalar, but ∇f is a vector!

Ex: $f(x,y) = x\cdot\cos(xy^2)$. $\nabla f = \langle \cos(xy^2) - xy^2\sin(xy^2), 2x^2y\sin(xy^2)\rangle$

$f_x = 1\cdot\cos(xy^2) + x\cdot(-\sin(xy^2))\cdot y^2$

$f_y = x(-\sin(xy^2))\cdot 2xy$

$f(x,y,z) = e^{x^2+y^2+z^2}$

$$\nabla f = \langle 2xe^{x^2+y^2+z^2}, 2ye^{x^2+y^2+z^2}, 2ze^{x^2+y^2+z^2}\rangle = f(x,y,z)\langle 2x, 2y, 2z\rangle$$

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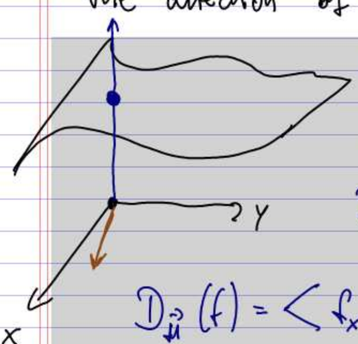
Panel 18

Ex: $f(x,y) = x^3 - 3xy + 4y^2$. Find directional derivative in the direction of $\langle\cos(\pi/6), \sin(\pi/6)\rangle$. at $(0,0)$



I stand at the dot on the surface, looking in $\langle\cos(\pi/6), \sin(\pi/6)\rangle$ direction. Do I have to go up or down?

$$D_{\vec{u}}(f) = \langle f_x, f_y\rangle\cdot\langle\cos(\pi/6), \sin(\pi/6)\rangle =$$

$$= \langle 3x^2 - 3y, -3x + 8y\rangle\cdot\langle\cos(\pi/6), \sin(\pi/6)\rangle = 0$$

$$\langle 0, 0\rangle\cdot\langle ?, ?\rangle = 0$$

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Panel 19

Note: $D_{\vec u} f = \nabla f \cdot \vec u$ $\qquad \frac{a \cdot b}{\|a\|\|b\|} = \cos(\alpha)$

$$\Rightarrow D_{\vec u} f = \nabla f \cdot \vec u = \|\nabla f\| \cdot \|\vec u\| \cos(\alpha)$$

$$\Rightarrow D_{\vec u} f = \|\nabla f\| \cos(\alpha), \quad \nabla f \text{ is fixed}$$

Useful Theorem: The max. value of $|D_{\vec u} f|$ is achieved if $\vec u$ points in the direction of ∇f. The maximum value is $\|\nabla f\|$.

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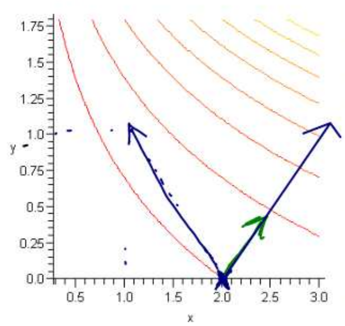
Panel 20

Ex: $f(x,y) = xe^y$. Then find rate of change at $P(2,0)$ in direction from P to $Q(1,1)$. In what direction does f have max. rate of change, and what is it?



Guess: in blue direction rate of change is less than in green direction!

$D_{\vec u}(f)$, $u = \vec{PQ} = \frac{1}{\sqrt 2} \langle -1, 1 \rangle$

$\nabla f = \langle e^y, xe^y \rangle$, $\nabla f \big|_{(2,0)} = \langle 1, 2 \rangle$

$$\Rightarrow D_{\vec u}(f) = \langle 1,2 \rangle \cdot \frac{1}{\sqrt 2} \langle -1, 1 \rangle = \langle 1, 2 \rangle \cdot \langle -\tfrac{1}{\sqrt 2}, \tfrac{1}{\sqrt 2} \rangle =$$

$$= -\tfrac{1}{\sqrt 2} + \sqrt 2 = 0.707$$

Direction of steepest increase: $\nabla f = \langle 1, 2 \rangle$

Max Rate of change $\|\nabla f\| = \sqrt 5 = 2.23$

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Panel 21

Ex: $f(x,y) = \sin(x)\cdot\cos(y)$ at $P(0,0)$

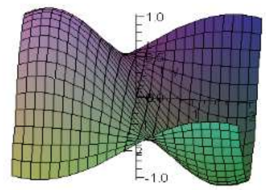
In which direction is the function the steepest? How steep is it in that direction?

$\Rightarrow$ Direction of steepest increase/decr. : ∇f

$$\Rightarrow \nabla f\Big|_{(0,0)} = \langle \cos(x)\cos(y), -\sin(y)\sin(x)\rangle\Big|_{(0,0)} = \langle 1, 0\rangle$$

Steepest increase is $\|\nabla f\|$

$\Rightarrow \|\nabla f\| = 1$



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