Panel 1

Ex: A particle starts at $P(1,0,0)$ with initial velocity $\langle 1,-1,1\rangle$. Acceleration is $\vec{a}(t) = \langle 4t, 6t, 1\rangle$.
Find its velocity and position function.

$$\vec{a} = v' = r'' = \langle 4t, 6t, 1\rangle$$

$$\Rightarrow v = \langle \underset{t=0}{2t^2} + c_1,\ \underset{t=0}{3t^2} + c_2,\ \underset{t=0}{t} + c_3 \rangle = \langle 2t^2+1,\ 3t^2-1,\ t+1\rangle$$

$$v(0) = \langle 1,-1,1\rangle = \langle c_1, c_2, c_3\rangle$$

$$\Rightarrow r(t) = \langle \frac{2}{3}t^3 + t + d_1,\ t^3 - t + d_2,\ \frac{1}{2}t^2 + t + d_3 \rangle$$

$d_1 = 1$
$d_2 = 0$
$d_3 = 0$

starting point $r(0)$

1

Panel 2

$$a(t) = \langle \cos(t), \sin(t)\rangle$$

initial vel. $\langle 0,0\rangle$
starting point $\langle 0,0\rangle$

$$v(t) = \langle \sin(t) + c_1,\ -\cos(t) + c_2 \rangle$$

$c_1 = 0$
$c_2 = +1$

2

Panel 3

**Quiz 4**

Suppose $\vec{r}(t) = \langle t^2, 2, t \rangle$ is a vector-valued function (aka space curve), representing the position of a particle. Find the following:

1. The velocity at $P(0,\cancel{1}\,2,0)$ ✓
2. The speed at $P(0,\cancel{1}\,2,0)$ ✓
3. The acceleration at $P(0,\cancel{1}\,2,0)$ ✓
4. The unit tangent $\vec{T}(t)$ at $P(0,\cancel{1}\,2,0)$ ✓
5. The unit normal vector $\vec{N}(t)$ at $P(0,\cancel{1}\,2,0)$ ✓ Sudo
6. The bi-normal vector $\vec{B}(t)$ at $P(0,\cancel{1}\,2,0)$ ✓
7. The curvature $k$ at $P(0,\cancel{1}\,2,0)$ ✓
8. The tangential component of the acceleration $a_T$ at $P(0,\cancel{1}\,2,0)$
9. The normal component of the acceleration $a_N$ at $P(0,\cancel{1}\,2,0)$
10. The osculating plane at $P(0,\cancel{0}\,2,0)$
11. The osculating circle at $P(0,\cancel{0}\,2,0)$

$$a_T = \frac{r' \cdot r''}{\|r'\|}$$

$$a_N = \frac{\|r' \times r''\|}{\|r'\|}$$

3

Panel 4

$r', r'', \|r'\|$

$r(t) = \langle t^2, 2, t \rangle \quad a_T, a_N$ at $P(0,2,0)$, i.e. $t = 0$

① $r'(t) = \langle 2t, 0, 1 \rangle \Rightarrow r'(0) = \langle 0, 0, 1 \rangle$

$r''(t) = \langle 2, 0, 0 \rangle \Rightarrow r''(0) = \langle 2, 0, 0 \rangle$

$$a_T = \frac{r' \cdot r''}{\|r'\|} = 0$$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} = \langle 0, 2, 0 \rangle$$

$$a_N = \frac{\|r' \times r''\|}{\|r'\|} = \frac{\|\langle 0,2,0 \rangle\|}{1} = \frac{2}{1} = \underline{\underline{2}}$$

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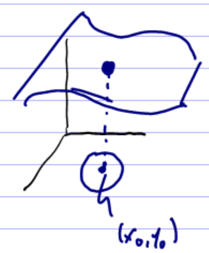
Panel 5

## Limits




Def: The limit of $f(x,y)$ as $(x,y)$ approaches $(x_0, y_0)$ is $L$, or

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{if}$$

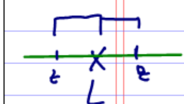
given any $\varepsilon > 0$ there is $\delta > 0$ such that

① if "$(x,y)$ is as close to $(x_0, y_0)$ as $\delta$"

② then "$f(x,y)$ is as close to $L$ as $\varepsilon$", i.e.

$$\text{if} \quad \|(x,y) - (x_0, y_0)\| = \underbrace{\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta}_{\text{circle centered at } (x_0,y_0)}$$

$$\text{then} \quad |f(x,y) - L| < \varepsilon$$




if $|L - \varepsilon| < \varepsilon$

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Panel 6

Helpful hint when limits do NOT exist:

Say $C_1$ is a path to $(x_0, y_0)$ and $f(x,y) \to L_1$ along $C_1$

$C_2$ is another path to $(x_0, y_0)$ and $f(x,y) \to L_2$ along $C_2$

Then, if $L_1 \neq L_2$ the limit does not exist.

Ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist because:

$x = 0$: $\displaystyle\lim \frac{-y^2}{y^2} = -1$

$y = 0$: $\displaystyle\lim \frac{x^2}{x^2} = 1$

$\Big\}$ different, i.e.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} \ \underline{\text{d.n.e.}}$$

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Panel 7

Ex: $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

$x=0$ : $\lim \frac{0}{y^2} = 0$

$y=0$ : $\lim \frac{0}{x^2} = 0$

$x=y$ : $\lim \frac{3x^3}{2x^2} = 0$

$x=y^2$ : $\lim_{y\to 0} \frac{3y^5}{y^4+y^2} = 0$

$y=x^2$ : $\lim_{x\to 0} \frac{3x^4}{x^2+x^4} = \lim_{x\to 0} \frac{12x^3}{2x+4x^3} = \lim \frac{36x^2}{2+12x^2} = 0$

This tells me NOTHING

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Panel 8

Prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

Have to show: take $\varepsilon > 0$, need to find $\delta > 0$

s.t. $\left| \underbrace{\frac{3x^2y}{x^2+y^2}}_{f(x,y)} - \overset{L}{0} \right| < \varepsilon$ if $\|(x,y)-(0,0)\| < \delta$

$\left| \frac{3x^2y}{x^2+y^2} \right| < \varepsilon$ if $\sqrt{x^2+y^2} < \delta$

$\frac{3x^2|y|}{x^2+y^2} < \varepsilon$ Know $\frac{x^2}{x^2+y^2} < 1$

$\Rightarrow \frac{3x^2|y|}{x^2+y^2} < 3|y| = \sqrt{y^2} \leq 3\sqrt{x^2+y^2}$

Proof: Given any $\varepsilon > 0$ pick $\delta = \varepsilon/3$ Then

8

Panel 9

pick $\delta = \varepsilon/3$

$\Rightarrow \sqrt{x^2+y^2} < \delta$

$\Rightarrow \sqrt{x^2+y^2} < \varepsilon/3$

$\Rightarrow 3\sqrt{x^2+y^2} < \varepsilon$

$\Rightarrow \frac{x^2|y|}{x^2+y^2} < \varepsilon$

$\Rightarrow |f(x,y) - 0| < \varepsilon$

quod erat demonstrandum

q.e.d.

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Panel 10

To tackle limits in $\mathbb{R}^2$ (or $\mathbb{R}^3$)

① Try the obvious (plug in)

② Try different approaches. Commonly used:

$x=0,\ y=0$

$x=y$

$x=y^2,\ y=x^2$

different DONE

③ Try to prove the limit is the common number found in step 2.

Give up unless you have time

Prove $f(x,y) \to L$

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Panel 11

Ex: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

$x=0$ : limit is $0$
$y=0$ : limit is $1$
} DNE ✓

Ex: $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$

$x=0$ : limit $0$ $\quad$ $y=x^2$ : $\lim \frac{2x^4}{2x^4} = 1$
$y=0$ : lim $0$

DNE

Ex: $\lim\limits_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$

$x=0$, $x=y$, $y=x^2$
$y=0$, $x=y^2$, anything
} $\lim = 0$

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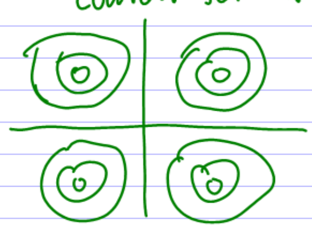
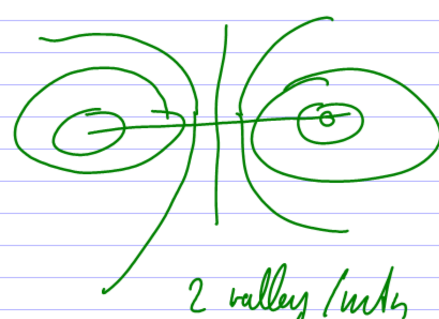
Panel 12

contour set A

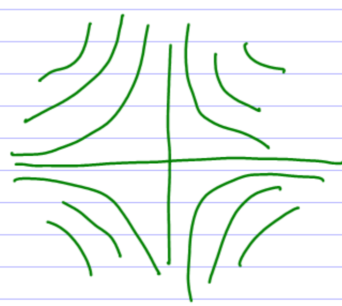


4 valley / mtn

contour set B



2 valley / mtn



For quiz:

$$a_T = \frac{r' \cdot r''}{\|r'\|}$$

$$a_N = \frac{\|r' \times r''\|}{\|r'\|}$$

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Panel 13

## Continuity:

As usual, continuity is basically a rephrased limit problem.

**Def:** $f(x,y)$ is continuous at $(x_0,y_0)$ if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

**Ex:** Is $f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ \cancel{2}\ 0 & \text{if } (x,y)=(0,0) \end{cases}$ continuous?

No!

What about $f(x,y) = \frac{3x^2y}{x^2+y^2}$ is clearly not cont. at $(0,0)$

$$f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & (x,y)\neq(0,0) \\ \cancel{2}\ 0 & (x,y)=(0,0) \end{cases} \Rightarrow \quad f(0,0) = \cancel{2}\ 0$$

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

before!

YES

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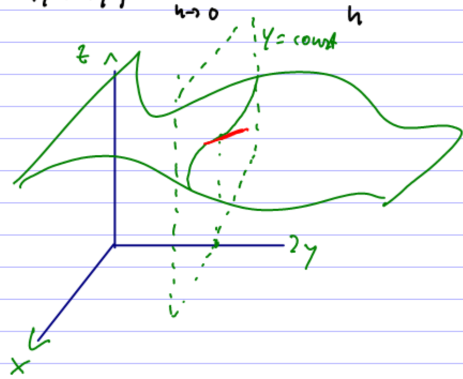
Panel 14

**Def:** If $f$ is a function of two variables, its partial derivatives are the functions $f_x$ and $f_y$ defined by.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \quad \text{partial with resp. to } x \ (y \text{ const})$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \quad \text{partial with resp. to } y \ (x \text{ const})$$




fix $y$ to take $f_x$

$f_x$ is slope of curve cut out by $y$ = const., or "slope in x-direction"

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Panel 15

**Ex:** Find $f_x$ for $f(x,y) = x^2 y + y^2$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{\left((x+h)^2 y + \cancel{y^2}\right) - \left(x^2 y + \cancel{y^2}\right)}{h} =$$

$$= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2)y - x^2 y}{h} = \lim_{h\to 0} \frac{y(\cancel{x^2} + 2xh + h^2 - \cancel{x^2})}{h} =$$

$$= \lim_{h\to 0} \frac{y\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h\to 0} y(2x+h) = 2xy$$

**How to Really find Partials**

To find $f_x$, think of $y$ as constant:

$f(x,y) = x^2 y + y^2$, $\Rightarrow f_x(x,y) = 2xy$

$f_y(x,y) = x^2 + 2y$ (check using limits)

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Panel 16

**Ex:** $f(x,y) = x^3 + x^2 y^3 - 2y^2$, find

$f_x(2,1)$: $f_x(x,y) = 3x^2 + 2xy^3$

$f_x(2,1) = 3\cdot 4 + 2\cdot 2\cdot 1^3 = 12 + 4 = 16$

If I stood at $f(2,1)$, looking in $x$+ direction, it goes up, and steep

$f_y(2,1)$: $f_y(x,y) = 3x^2y^2 - 4y$

$f_x(2,1) = 12 - 4 = 8$

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Panel 17

Of course we can also take higher-order partials.

$f(x,y)$ a function of two variables

1st order: $f_x$ $f_y$

2nd order: $f_{xx}$ $f_{xy} \xleftrightarrow[\text{same}]{\text{usually}} f_{yx}$ $f_{yy}$

3rd order: $f_{xxx}$ $f_{xxy}$ $f_{xyx}$ $f_{xyy}$ $f_{yxx}$ $f_{yxy}$ $f_{yyx}$ $f_{yyy}$

1 2 3 4

Other notation: $f_x = \frac{\partial f}{\partial x}$ $f_y = \frac{\partial f}{\partial y}$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$ $f_{xyx} = \frac{\partial^3 f}{\partial x \partial y \partial x}$

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Panel 18

Ex: $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x(x,y) = 3x^2 + 2xy^3$

$f_y(x,y) = 3x^2y^2 - 4y$

$f_{xx}(x,y) = 6x + 2y^3$

$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 + 2xy^3\right) = 6xy^2$

$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x^2y^2 - 4y\right) = 6xy^2$

$f_{yx} = (f_y)_x$

$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(3x^2y^2 - 4y\right) = 6x^2y - 4$

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Panel 19

$$f(x,y) = (x^2+y^2)\sin(xy)$$

Find $f_{xy}$

$$f_x(x,y) = 2x\sin(xy) + (x^2+y^2)\cos(xy)\cdot y$$

$$f_{xy}(x,y) = 2x\cos(xy)\cdot x$$

$$2y\cos(xy)\cdot y + (x^2+y^2)\left(-\sin(xy)xy + \cos(xy)\right)$$

$f_{yy}$:

$$f_y = 2y\sin(xy) + (x^2+y^2)\cos(xy)\cdot x$$

$$f_{yy} = 2\sin(xy) + 2y\cos(xy)\cdot x +$$

$$2y\cos(xy)\cdot x + (x^2+y^2)\left(-\sin(xy)x^2\right)$$

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Panel 20

Other notation: $f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f$

$f_{xy} = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2}{\partial x\, \partial y} f$ etc.

Done already

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Panel 21

Partial derivatives frequently occur in partial differential equations that express physical laws.

Laplace PDE: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\|$

Important in heat conduction, fluid flow, electric potential

Ex: $f(x,y) = e^x \cdot \sin(y)$ is a solution to the Laplace equation.

$f_x = e^x \sin(y) \Rightarrow f_{xx} = e^x \sin(y)$

$f_y = e^x \cdot \cos(y) \Rightarrow f_{yy} = -e^x \sin(y)$

$\Rightarrow f_{xx} + f_{yy} = 0$

q.e.d.

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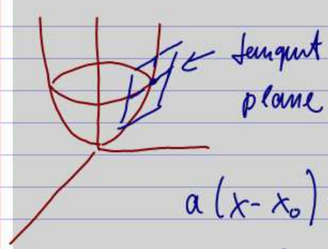
Panel 22

$f_x$ - tangent in x-direction
$f_y$ - tangent in y-direction
} $f_x$ and $f_y$ determine the tangent plane!

Find equation of tangent plane to $f(x,y)$ at $(x_0, y_0)$.




Want: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Can't cross-product $f_x \times f_y$ !

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \quad \Big| : -c$

$a'(x-x_0) + b'(y-y_0) + z_0 = z = g(x,y)$

$g_x = a'$ Thus, the tangent plane to $f(x,y)$

$g_y = b'$ must have $a' = f_x$, $b' = f_y$

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Panel 23

Equation of tangent plane to $f(x,y)$ at $(x_0,y_0)$ is:

$$\boxed{z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0}$$

Ex: $f(x,y) = 2x^2 + y^2$. Find tangent plane to $f$ at $P(1,1,3)$.

$P(1,1,3)$ : $x=1$, $y=1$ $\Rightarrow z = 3$ ✓

$f_x = 4x \Rightarrow f_x(x_0,y_0) = 4$

$f_y = 2y \Rightarrow f_y(x_0,y_0) = 2$

$\Rightarrow z = 4\cdot(x-1) + 2(y-1) + 3$

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Panel 24

Review: $r(t) = \langle t, \sin(t), \cos(t) \rangle$

equation of tangent plane at $\langle 0,0,1 \rangle$ ($t=$

$r'(t) = \langle 1, \cos(t), -\sin(t) \rangle$

$\Rightarrow r'(0) = \boxed{\langle 1,1,0 \rangle}$

$N = T'/\|T'\|$ : $T = \frac{1}{\sqrt{2}} \langle 1, \cos(t), -\sin(t) \rangle$

$\Rightarrow T' = \langle \#, \#, \# \rangle \sim N$

$B = T \times N = \langle a, b, c \rangle$

$\Rightarrow a(x- \ ) + b(y- \ ) + c(z- \ ) = 0$

$B = T \times N$ defines plane

$f(x,y) = e^x \sin(y)$ at $(0,0,0)$

$f_x$ ....

$f_y$ .....

$z = f_x(x- \ ) + f_y(y- \ ) + 0$

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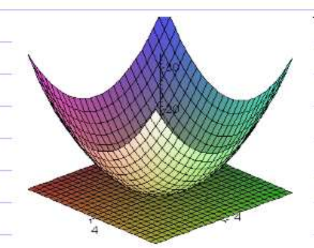
Panel 25

25

Panel 26

Find and interpret the tangent plane to

$f(x,y) = x^2+y^2+1$ at $(0,0)$



$f_x = 2x \Rightarrow f_x(0,0) = 0$

$f_y = 2y \rightarrow f_y(0,0) = 0$

$\Rightarrow z = f_x\,(x- \quad) + f_y\,(y- \quad) + z_0 =$

$= 1$

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