Panel 1

Last time:

planes

distance

vector-valued functions, space curve

limits, deriv., int.

length of a curve

Warm-Up: $l(t) = \langle 1+t, 2-t, 3t \rangle$, $x+4y+z=2$

Are they parallel? If so find distance.

normal to plane is $\langle 1,4,1 \rangle$

dir. of line is $\langle 1,-1,3 \rangle$

dot: $\langle 1,4,1 \rangle \cdot \langle 1,-1,3 \rangle = 0$

1

Panel 2

Distance $l(t) = \langle 1+t, 2-t, 3t \rangle$, $x+4y+z=2$

Know distance not zero

point on line: $P(1,2,0)$ $(t=0)$

point on plane: $Q(0,0,2)$

$\Rightarrow$

$$\text{proj}_{\vec{n}}(PQ) = \frac{PQ \cdot \vec{n}}{\|\vec{n}\|} \frac{n}{\|n\|}$$

length 1

$$\text{length}(\text{proj}_{\vec{n}}(PQ)) = \frac{|PQ \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|\langle -1,-2,+2 \rangle \langle 1,4,1 \rangle|}{\sqrt{18}} =$$

$$= \left| \frac{-7}{\sqrt{18}} \right| = 7/\sqrt{18}$$

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Panel 3

Ex: Find length of $r(t) = \langle \overset{x}{\cos(t)}, \overset{y}{\sin(t)}, t \rangle$ from $(1,0,0)$ to $(1,0,2\pi)$.

$$L = \int_a^b \|r'(t)\|\,dt$$

$$r(t) = \langle \cos(t), \sin(t), t \rangle \qquad t = 0 \text{ to } 2\pi$$

$$r'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$\|r'(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

$$\Rightarrow L = \int_0^{2\pi} \sqrt{2}\,dt = \boxed{2\pi\sqrt{2}}$$

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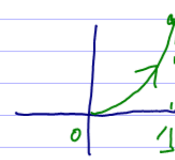
Panel 4

It is entirely possible for one curve to have many different parametrizations:

Ex: $r_1(t) = \langle \underset{x}{t}, \underset{y}{t^2} \rangle$, $t = 0, 1$

$y = t^2 = x^2$



Same as: $r(t) = \langle t^3, t^6 \rangle$

$x = t^3$, $y = t^6 = (t^3)^2 = x^2$

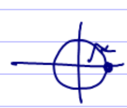
different speed!

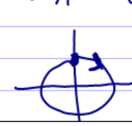
$\langle \cos(t), \sin(t) \rangle \quad t = 0, 2\pi$

$\langle \underset{x}{\cos(2t)}, \underset{y}{\sin(2t)} \rangle, \quad t = 0, \pi$

$t = 0, 2\pi$

$\langle \sin(t), \cos(t) \rangle$





write $y = e^t/x$ as space curve:

$r(t) = \langle t, \frac{e^t}{t} \rangle$

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Panel 5

Ex. Compute length of $r(t) = \langle t, \sqrt{1-t^2} \rangle$, $t = -1, 1$

$$L = \int_{-1}^{1} \sqrt{1^2 + \left(\frac{-2t}{2\sqrt{1-t^2}} \right)^2}\, dt =$$

$$= \int_{-1}^{1} \sqrt{1 + \frac{t^2}{1-t^2}}\, dt = \int_{-1}^{1} \sqrt{\frac{1-t^2+t^2}{1-t^2}}\, dt = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, dt$$

$$= \arcsin(t) \Big|_{-1}^{1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

circle: $\langle \cos(t), \sin(t) \rangle$, $t = \pi$ to 0

$$\Rightarrow L = \int_0^{\pi} \sqrt{1}\, dt = \underline{\pi}$$

$\langle t, \sqrt{1-t^2} \rangle$, $t = -1, 1$

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Panel 6

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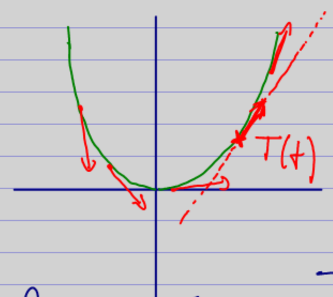
Panel 7

Def: A curve $r(t)$ is called <u>smooth</u> if $\vec{r}\,'(t) \neq \vec{0}$, i.e. if the components of r' are not simultaneously zero.

Def. If $r(t)$ is a smooth curve then

$T(t) = r'(t) / \|r'(t)\|$ is <u>unit</u> tangent at t

Ex: $r(t) = \langle t, t^2 \rangle$ find $T(t)$.



$$T(t) = r'(t) / \|r'\|$$

$$r'(t) = \langle 1, 2t \rangle$$

at $t=0$: $T = \langle 1, 0 \rangle$ $\quad T(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle$

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Panel 8

We can now measure direction of curve (derivative) and length (arc length). Next we measure

Def: The <u>curvature</u> measures how fast the tangent changes. It is defined as

$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|}$$ where T is unit tangent.

circles, radius r

Ex: $\vec{r}(t) = \langle r\cos(t), r\sin(t) \rangle$ Find κ (curvature)

$r'(t) = \langle -r\sin(t), r\cos(t) \rangle$

$\Rightarrow T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{r} \langle -r\sin(t), r\cos(t) \rangle$

$= \langle -\sin(t), \cos(t) \rangle$

$T'(t) = \langle -\cos(t), -\sin(t) \rangle$

$\Rightarrow \kappa = \frac{\|T'\|}{\|r'\|} = \frac{1}{r}$

constant curvature

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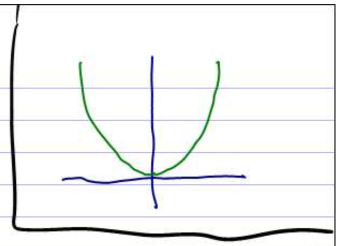
Panel 9

Ex: Find curvature of $r(t) = \langle t, t^2 \rangle$



$$\kappa = \frac{\|T'\|}{\|r'\|}, \quad T = r'/\|r'\|$$

$$r'(t) = \langle 1, 2t \rangle, \quad \|r'(t)\| = \sqrt{1+4t^2}$$

$$T(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle = \left\langle \frac{1}{(1+4t^2)^{1/2}}, \frac{2t}{(1+4t^2)^{1/2}} \right\rangle$$

$$T'(t) = \left\langle -\frac{1}{2}(1+4t^2)^{-3/2} \cdot (8t), \frac{2(1+4t^2)^{1/2} - 2t\frac{1}{2}(1+4t^2)^{-1/2} \cdot 8t}{[(1+4t^2)^{1/2}]^2} \right\rangle =$$

$$= \left\langle \frac{-4t}{(1+4t^2)^{3/2}}, \frac{(1+4t^2)^{-1/2}(2(1+4t^2) - 8t^2}{(1+4t^2)} \right\rangle$$

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Panel 10

$$T'(t) = \left\langle -\frac{1}{2}(1+4t^2)^{-3/2} \cdot (8t), \frac{2(1+4t^2)^{1/2} - 2t\frac{1}{2}(1+4t^2)^{-1/2} \cdot 8t}{[(1+4t^2)^{1/2}]^2} \right\rangle =$$

$$= \left\langle \frac{-4t}{(1+4t^2)^{3/2}}, \frac{(1+4t^2)^{-1/2}(2(1+4t^2) - 8t^2}{(1+4t^2)} \right\rangle =$$

$$= \left\langle \frac{-4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle = \frac{1}{(1+4t^2)^{3/2}} \langle -4t, 2 \rangle$$

$$\|T'(t)\| = \frac{1}{(1+4t^2)^{3/2}} \sqrt{4+16t^2} = \frac{2}{(1+4t^2)^{3/2}} \sqrt{1+4t^2} = 2/(1+4t^2)$$

$$\kappa = \frac{\|T'(t)\|}{\|r'\|} = 2/(1+4t^2) \cdot \frac{1}{\sqrt{1+4t^2}} = 2/(1+4t^2)^{3/2}$$

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Panel 11

Thm The curvature of $\vec{r}(t)$ is

$$\kappa(t) = \frac{\|r'(t)\times r''(t)\|}{\|r'(t)\|^3}$$

much better is this

Ex: $\vec{r}(t) = \langle t, t^2 \rangle = \langle t, t^2, 0 \rangle$

Need: $r' = \langle 1, 2t, 0 \rangle$

$r'' = \langle 0, 2, 0 \rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

$$\Rightarrow \kappa = \frac{|\langle 0,0,2\rangle|}{\left(\sqrt{1+4t^2}\right)^3} = \frac{2}{(1+4t^2)^{3/2}}$$

back at 9:50

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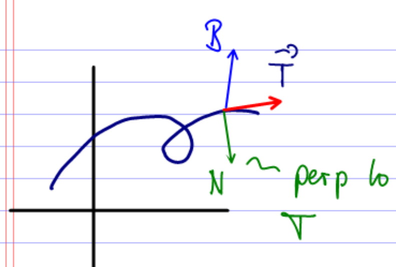
Panel 12



Thm: If $\|\vec{r}\| = 1$ then

$r \cdot r' = 0$

Proof: $\|r\|^2 = r \cdot r$

$$0 = \frac{d}{dt}(1) = \frac{d}{dt}\left(\|r\|^2\right) = \frac{d}{dt}\, r \cdot r = r' \cdot r + r \cdot r' = 2\boxed{r \cdot r'}$$

$\Rightarrow r \cdot r' = 0$

Def: If $T(t)$ is unit tangent vector to $\vec{r}(t)$ then $T'(t)$ is perp. to T (by above thm). We call

$N(t) = T'(t) / \|T'(t)\|$ principle unit normal vector

$B(t) = T \times N$ binormal vector

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Panel 13

Ex: Suppose $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find unit tangent, unit normal, and binormal vectors

$r(t) = \langle \cos(t), \sin(t), t \rangle$

$r'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

$T(t) = \frac{1}{\sqrt{2}} \cdot \langle -\sin(t), \cos(t), 1 \rangle$

usually hard

$T'(t) = \boxed{\frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle}$ $\quad \|T'\| = \frac{1}{\sqrt{2}} \cdot 1$

$N(t) = \frac{T'}{\|T'\|} = \frac{1}{\cancel{\sqrt{2}}} \langle -\cos(t), -\sin(t), 0 \rangle \cdot \cancel{\sqrt{2}} =$

$= \langle -\cos(t), -\sin(t), 0 \rangle$

$B(t) = T \times N = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle \times \langle -\cos(t), -\sin(t), 0 \rangle$

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Panel 14

$B(t) = T \times N = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle \times \langle -\cos(t), -\sin(t), 0 \rangle$

$=$

$$\begin{vmatrix} i & j & k \\ -\sin(t) & \cos(t) & 1 \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} = \langle \sin(t), \cos(t), 1 \rangle$$

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Panel 15

Summary:

$\vec{r}(t)$ space curve

$T(t) = \frac{r'(t)}{\|r'(t)\|}$ (unit tangent)

$N(t) = \frac{T'(t)}{\|T'(t)\|}$ (principle unit normal)

$B(t) = T(t) \times N(t)$ (binormal)

$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$ (curvature)

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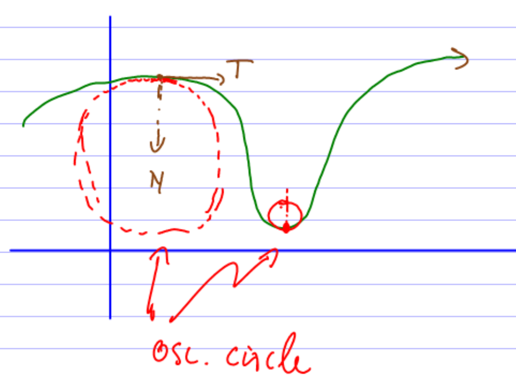
Panel 16

Def: The plane determined by T and N is called osculating plane or supporting plane. (Latin: osculum = kiss)

Def: The circle in the osculating plane with radius $r = 1/\kappa$ is called the osculating circle.



big curv = small circle

$1/\kappa$ = radius

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Panel 17

Suppose $u = <7,-2,3>$, $v = <-1,4,5>$, and $w = <-2,1,-3>$

a) Are *u* and *v* orthogonal, parallel, or neither?

b) Find the angle between *v* and *w*

c) Find $u \cdot v$ (dot product), $u \times v$ (cross product), $u \cdot (v \times w)$, and $\|u\|$

d) Find the equation of the line in the direction of *u* that goes through P(2, 0, 1)

e) Find the equation of the plane spanned by *v* and *w* through the origin

f) Find the equation of the plane spanned by *u* and *v* through the point P(2, 0, 1)

e) $v = \langle -1, 4, 5 \rangle$

$w = \langle -2, 1, -3 \rangle$

$$v \times w = \begin{vmatrix} i & j & k \\ -1 & 4 & 5 \\ -2 & 1 & -3 \end{vmatrix} = \langle -17, -13, 7 \rangle$$

$\Rightarrow \quad ax + by + cz = d$

$-17x - 13y + 7z = d$

Through $(0, 0, 0) \Rightarrow d = 0$

f) $(2, 0, 1)$ or $-17(x-2) - 13(y-0) + 7(z-1) = 0$

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Panel 18

5c a) If $r(t) = <4t, t^2, t^3>$, find $r'(t)$, $r''(t)$, $\frac{d}{dt}\|r(t)\|$, and $\left\|\frac{d}{dt} r(t)\right\|$

$r'(t) = \langle 4, 2t, 3t^2 \rangle$

$r''(t) = \langle 0, 2, 6t \rangle$

$\frac{d}{dt}\|r(t)\| = \frac{d}{dt}\left(\sqrt{16t^2 + t^4 + t^6}\right) = \frac{1}{2}(\quad)^{-1/2} \cdot (\quad)$

$\left\|\frac{d}{dt} r(t)\right\| = \|\langle 4, 2t, 3t^2 \rangle\| =$

$= \sqrt{16 + 4t^2 + 9t^4}$

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Panel 19

$(t, \frac{1}{t}) \Rightarrow y = \frac{1}{x}$, $t > 0$

~~a)~~ If $r(t) = \langle t, \frac{1}{t} \rangle$, find $T(t)$, $N(t)$, $B(t)$, *curvature*

$r(t) = \langle t, \frac{1}{t} \rangle$, $r'(t) = \langle 1, -\frac{1}{t^2} \rangle$

$T = r' / \|r'\| = \frac{1}{\sqrt{1 + \frac{1}{t^4}}} \langle 1, -\frac{1}{t^2} \rangle = \underbrace{\frac{t^2}{\sqrt{t^4+1}}} \underbrace{\langle 1, -\frac{1}{t^2} \rangle}$

$\underline{N} = T' / \|T'\| = \pm \frac{t^2}{\sqrt{t^4+1}} \langle \frac{1}{t^2}, 1 \rangle$

$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3}$ easy if $r = \langle t, \frac{1}{t}, \underline{0} \rangle$

$B = T \times N = HW$

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