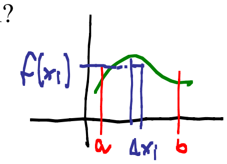
Practice Exam 3

Consider a definite integral of the form $\int_a^b f(x)dx$

a) What is the exact mathematical definition of that definite integral?

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} f(x_1)\Delta x_1 + \dots + f(x_n)\Delta x_n$$



b) What is the geometric interpretation of that definite integral?

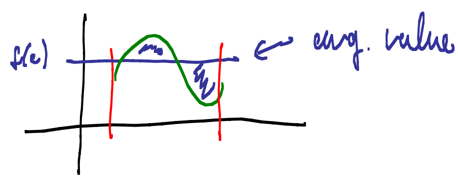
$\int_a^b f(x)dx$ is the area under curve $f(x)$ if $f(x)$ is positive, otherwise it is the "signed" or "net" area

c) What is the definition of the average value of a function over the interval [a, b]

$$\text{avg.} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

d) What is the geometric interpretation of the average value of a function over the interval [a, b]

horizontal line where graph $f(x)$ would balance



e) What is the MVT for Integrals

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c) \text{ for some } c \text{ in } [a,b]$$

f) State the first fundamental theorem of calculus, as it applies to that integral. What is that theorem good for, in your own words?

(1) $\int_a^b f(x)dx = F(b) - F(a)$, F is antiderivative of f. Used to evaluate integrals

(2) $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, used to define new functions

g) What is the definition of the natural log function *ln(x)*

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

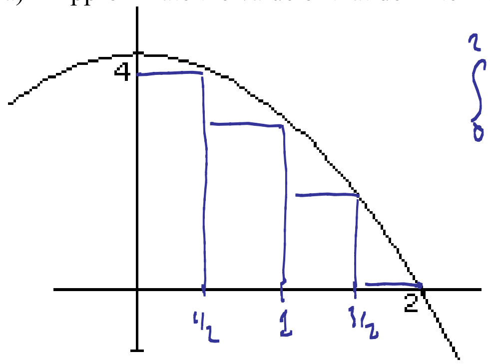
h) What is the difference between $\int_a^b f(x)dx$ and $\int f(x)dx$

$\int_a^b f(x)dx$ gives a number (definite integral)

$\int f(x)dx$ gives a function (indefinite integral)

Consider the following definite integral: $\int_0^2 4 - x^2 dx$ (The integrand is depicted below).

a) Approximate the value of that definite integral by using 4 subdivisions and <u>right</u> rectangles in the corresponding summation.



$$\int_0^2 4-x^2\,dx \approx f(\tfrac12)\cdot\tfrac12 + f(1)\tfrac12 + f(\tfrac32)\cdot\tfrac12 + f(2)\tfrac12 =$$

$$= \tfrac12\left((4-\tfrac14)+(4-1)+(4-\tfrac94)+(4-4)\right) =$$

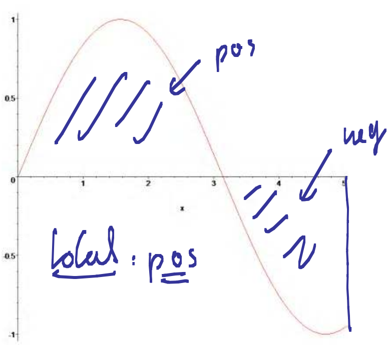
$$= \tfrac12\left(12 - 1 - \tfrac{10}{4}\right) = \tfrac12\cdot\tfrac{34}{4} = \tfrac{17}{4} = 4.25$$

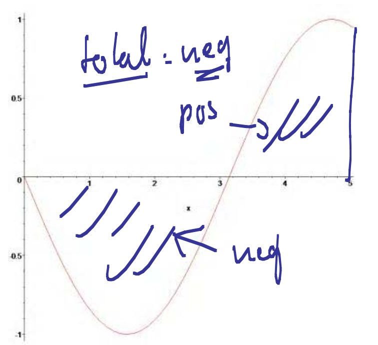
b) Find the exact value of that definite integral by using the first fundamental theorem of calculus.

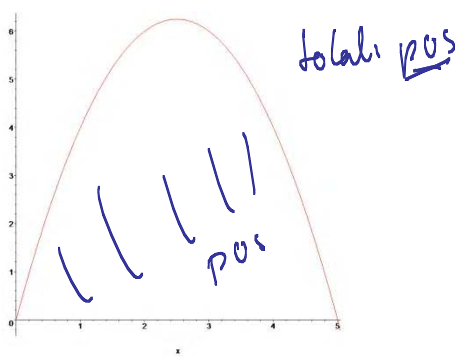
$$\int_0^2 4-x^2\,dx = 4x - \tfrac13 x^3\Big|_0^2 = \left(4\cdot2-\tfrac13 2^3\right)-\left(4\cdot0-\tfrac13 0^3\right) =$$

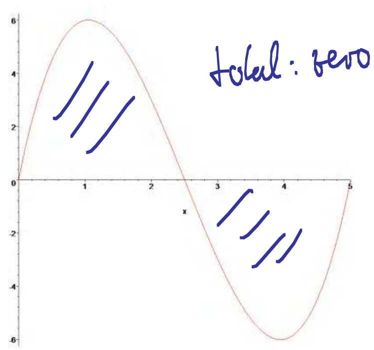
$$= 8-\tfrac83 = \tfrac{16}{3} = 5.33$$

Below are the graphs of three functions. For each graph, decide whether $\int_0^5 f(x)dx$ is positive, negative, or zero.









Evaluate each of the following integrals. If you use substitution, please indicate clearly what your ***u*** is, and what the corresponding ***du*** is.

$$\int 3x^2 - \frac{5}{x^3} + \sqrt[3]{x} - 2\cos(x)dx = \int 3x^2 - 5x^{-3} + x^{1/3} - 2\cos(x)\,dx =$$

$$= x^3 - 5\cdot\left(-\tfrac{1}{2}\right)x^{-2} + \tfrac{3}{4}x^{4/3} - 2\sin(x) + C$$

$$\int x^2 + \frac{1}{\sqrt{x}} + \pi^2 dx$$

$$= \int x^2 + x^{-1/2} + \pi^2 dx = \tfrac{1}{3}x^3 + 2x^{1/2} + \pi^2 x + C$$

$$\int \frac{\overbrace{3x^2 dx}^{du}}{\sqrt{\underbrace{x^3+1}_{u}}} = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}du = 2u^{1/2} + C = 2(x^3+1)^{1/2} + C$$

$$\boxed{\begin{aligned} u &= x^3+1 \\ du &= 3x^2\,dx \end{aligned}}$$

$$\int \frac{2}{x}dx = 2\ln(x) + C$$

$$\int \tan(x)dx = \int \frac{\overbrace{\sin(x)\,dx}^{-du}}{\underbrace{\cos(x)}_{u}} = -\int \frac{1}{u}du = -\ln(u) + C = -\ln(\cos(x)) + C$$

$$= \ln(u^{-1}) + C = \ln(\sec(x)) + C$$

either one is right

$u = \cos(x)$
$du = -\sin(x)dx$

$$\int x\sqrt{x-1}dx$$

tricky because subst does not seem to work.

$u = x-1 \Rightarrow x = u-1$!
$du = dx$

$$\int x\sqrt{x-1}\,dx = \int (u-1)\sqrt{u}\,du$$

$$= \int u^{3/2} - u^{1/2}du =$$

$$= \tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2} + C$$

$$= \tfrac{2}{5}(x-1)^{5/2} - \tfrac{2}{3}(x-1)^{3/2} + C$$

$$\int 5x\sqrt{x^2+1}dx = 5\int \overbrace{x\sqrt{\underbrace{x^2+1}_{u}}\,dx}^{1/2\,du} =$$

$u = x^2+1$
$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$$\tfrac{5}{2}\int\sqrt{u}\,du = \tfrac{5}{\cancel{2}}\cdot\tfrac{\cancel{2}}{3}u^{3/2} + C =$$

$$= \tfrac{5}{3}(x^2+1)^{3/2} + C$$

$$\int \frac{\overbrace{\sin(x)\,dx}^{-du}}{\cos^3(x)} = -\int \frac{1}{u^3}du =$$

$u = \cos(x)$
$du = -\sin(x)dx$

$$= -\left(-\tfrac{1}{2}\right)u^{-2} + C = \tfrac{1}{2}(\cos(x))^{-2} + C = \tfrac{1}{2}\sec^2(x) + C$$

either way is fine!

$\int \frac{\cos(2x)}{\sin^3(2x)}dx$ (Hint: use $\boldsymbol{u = \sin(2x)}$) $-\frac{1}{2}du$

$u = \sin(2x)$

$\Rightarrow du = \cos(2x)\,2\,dx$

$\Rightarrow \frac{1}{2}du = \cos(2x)dx$

$\int \frac{\cos(2x)}{\sin^3(2x)}dx = \frac{1}{2}\int \frac{1}{u^3}du =$

$= \frac{1}{2}\left(-\frac{1}{2}\right)u^{-2} + C = -\frac{1}{4}\left(\sin(2x)\right)^{-2} + C$

$\int \frac{1}{x\ln(x)}dx$

$u = \ln(x)$

$du = \frac{1}{x}dx$

$\int \frac{1}{x\ln(x)}dx = \int \frac{1}{u}du = \ln|u| + C = \ln(\ln(x)) + C$

$\int (x+\frac{1}{x})^2 dx = \int x^2 + 2x\cdot\frac{1}{x} + (\frac{1}{x})^2 dx = \int x^2 + 2 + x^{-2}dx =$

no subst. just foil

$= \frac{1}{3}x^3 + 2x - x^{-1} + C$

Evaluate each of the following definite integrals.

$\int_0^{\pi/2}\cos(x)dx = \sin(x)\Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1$

$\int_1^2 x(x-1)dx = \int_1^2 x^2 - x\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2\Big|_1^2 =$

$\left(\frac{1}{3}2^3 - \frac{1}{2}2^2\right) - \left(\frac{1}{3}1^3 - \frac{1}{2}\cdot 1^2\right) =$

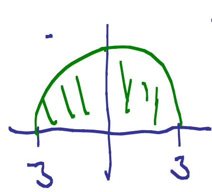
$8/3 - 2 - 1/3 + 1/2 = 7/3 - 2 + 1/2 = 5/6$

$\int_{-3}^{3}\sqrt{9-x^2}dx$ (trick question)

can't find antiderivative, but solve it graphically

$y = \sqrt{9-x^2} \Rightarrow y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9$

thus, integral is 1/2 area of circle



$A = \frac{9\pi}{2}$

$\int_1^4 \frac{x-1}{\sqrt{x}}dx = \int_1^4 \frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}dx = \int_1^4 x^{1/2} - x^{-1/2}dx =$

$= \frac{2}{3}x^{3/2} - 2x^{1/2}\Big|_1^4 =$

$= \left(\frac{2}{3}4^{3/2} - 2\cdot 4^{1/2}\right) - \left(\frac{2}{3}1 - 2\cdot 1\right) =$

$16/3 - 4 - 2/3 + 2 = \frac{14}{3} - 2 = \frac{8}{3}$

$\int_1^2 \frac{2-t^3}{t^2}dt = \int_1^2 \frac{2}{t^2} - \frac{t^3}{t^2} dt = \int_1^2 \frac{2}{t^2} - t \, dt = -2t^{-1} - \frac{1}{2}t^2 \Big|_1^2 =$

$= \left(-2 \cdot 2^{-1} - \frac{1}{2} 2^2\right) - \left(-2 - \frac{1}{2}\right) =$

$= -1 - 2 + 2 + \frac{1}{2} = -\frac{1}{2}$

$\int_{-1}^{1} \frac{x}{\sqrt{x^2+2}}dx = \int_{x=-1}^{x=1} \frac{1}{\sqrt{u}} du = 2u^{1/2}\Big|_{x=-1}^{x=1} = 2\sqrt{x^2+2}\Big|_{x=-1}^{x=1} =$

$= 2\sqrt{3} - 2\sqrt{3} = 0$

$u = x^2 + 2$

$du = 2x\,dx$

or $\frac{1}{2}du = x\,dx$

$\int_\pi^\pi \cos(4x^2)dx$ (trick question) $= 0$ because $\int_a^a f(x)\,dx = 0$ always

(same integration bounds)

$\int_{-1}^{1} \frac{1}{x+2}dx = \ln(x+2)\Big|_{-1}^{1} = \ln(3) - \ln(1) = \ln(3)$

hard $\int_0^1 \frac{x-1}{x+1} = \int_0^1 \frac{x}{x+1} - \frac{1}{x+1} dx = \overset{1-\ln(2)}{\int_0^1 \frac{x}{x+1} dx} - \int_0^1 \frac{1}{x+1} dx$ $= \ln(x+1)\Big|_0^1 = \ln(2) - \cancel{\ln(1)}$

$= 1 - \ln(2) - \ln(2) = 1 - 2\ln(2)$ final answer

$\int_0^1 \frac{x}{x+1}dx = \int_{x=0}^{x=1} \frac{u-1}{u} du = \int_{x=0}^{x=1} 1 - \frac{1}{u} du = u - \ln|u|\Big|_{x=0}^{x=1} = (x+1) - \ln(x+1)\Big|_0^1 =$

$= 2 - \ln(2) - 1 + \cancel{\ln(1)}^{0} =$

$= 1 - \ln(2)$

$u = x + 1 \iff u - 1 = x$

$du = dx$

At SLAC (Stanford Linear Accelerator) the initial position of a particle was recorded at time $t = 0$ to be 10m. Several detectors were used to record the speed of the particle, and it could be determined that the velocity function of the particle was given by $v(t) = 4t^2 + 3t$. What is the distance function of the particle, and where is the particle after 5 seconds?

$v(t) = 4t^2 + 3t$. $s(0) = 10$ meter

$\Rightarrow s(t) = \int 4t^2 + 3t\,dt = \frac{4}{3}t^2 + \frac{3}{2}t^2 + C$ Since $s(0) = 10$, $C = 10$

$\Rightarrow \boxed{s(t) = \frac{4}{3}t^2 + \frac{3}{2}t^2 + 10}$, $s(5) = \frac{1285}{6} = 214.16$

The probability that a person will remember between a% and b% of material learned in a certain experiment is $P_{a,b} = \int_a^b \frac{15}{4} x\sqrt{1-x}\,dx$.

For a randomly chosen individual, what is the probability that he or she will recall between 50% and 70% of the material?

$$\text{Answer} = \int_{0.50}^{0.70} \frac{15}{4} x\sqrt{1-x}\,dx = \frac{15}{4}\int_{0.50}^{0.70} x\sqrt{1-x}\,dx = -\frac{15}{4}\int_{x=0.5}^{x=0.7} (1-u)\sqrt{u}\,du =$$

$u = 1-x$ so $x = 1-u$

$du = -dx$

$$= -\frac{15}{4}\int_{x=0.5}^{x=0.7} u^{1/2} - u^{3/2}\,du$$

$$= 0.35309 \text{ (using Maple)}$$

Suppose that gasoline is increasing in price according to the equation $p = 1.2 + 0.04t$ where p is the dollar price per gallon, and t is the time in years, with $t = 0$ representing 1990. If an automobile is driven 15,000 miles a year, and gets M miles per gallon, the annual fuel cost is $C = \frac{15000}{M}\int_t^{t+1} p\,dt$. Estimate the annual fuel cost (a) for the year 2000 and (b) for the year 2005.

$$C = \frac{15000}{M}\int_t^{t+1} 1.2 + 0.04t \; :$$

constant

(a) $\frac{15000}{M}\int_{10}^{11} 1.2 + 0.04t\,dt =$

(b) $\frac{15000}{M}\int_{15}^{16} 1.2 + 0.04t\,dt =$

use Maple

Define a function $S(x) = \int_0^x \cos^2(t^2)\,dt$. Then find S(0), S'(x), and S''(x). Where is the function increasing and decreasing?

$S(0) = \int_0^0 \cos^2(t^2)\,dt = 0$ (same bounds of integration)

$S'(x) = \cos^2(x^2)$

$S''(x) = \underline{2\cos(x^2)(-\sin(x^2))\cdot(-2x)}$

incr. / decr. is too much work, sorry!

If $G(x) = \int_x^{x^2} \sin(t^2)\,dt$ then find G(1) and G'(x).

$G(1) = \int_1^{1^2} \sin(t^2)\,dt = \int_1^1 \sin(t^2)\,dt = 0$ (same bounds)

$G'(x) = \sin((x^2)^2)\,2x - \sin(x^2) =$

$= \underline{\sin(x^4)\,2x - \sin(x^2)}$

Find the derivatives of the following functions:

$$f(x)=\ln\left(\frac{\sqrt{x-1}}{(x-1)^2}\right)$$

$$f'(x)=\frac{1}{\frac{\sqrt{x-1}}{(x-1)^2}}\cdot\frac{\frac{1}{2}(x-1)^{-1/2}(x-1)^2-(x-1)^{1/2}\,2(x-1)}{(x-1)^4}=$$

$$=\frac{\cancel{(x-1)^2}}{\cancel{\sqrt{x-1}}}\cdot\frac{\cancel{(x-1)^{1/2}}\left[\frac{1}{2}(x-1)^{-1+2}-2(x-1)\right]}{(x-1)^{\cancel{4}\,2}}$$

$$=\frac{\frac{1}{2}(x-1)-2(x-1)}{(x-1)^2}=-\frac{3}{2}(x-1)^{-1}$$

Better

good

~~$f(x)=(x-1)^2\ln\left(\sqrt{x^2-1}\right)$~~

Easier:
$$\ln\left(\frac{\sqrt{x-1}}{(x-1)^2}\right)=\ln\left(\sqrt{x-1}\right)-\ln\left((x-1)^2\right)=\frac{1}{2}\ln(x-1)-2\ln(x-1)=$$

$$=-\frac{3}{2}\ln(x-1)\Rightarrow\frac{d}{dx}\left[\ln\left(\frac{\sqrt{x-1}}{(x-1)^2}\right)\right]=\frac{d}{dx}\left(-\frac{3}{2}\ln(x-1)\right)=$$

$$=-3/2\,(x-1)^{-1}$$

~~$f(x)=\frac{(x-1)^2}{(x+1)^3}(x+2)^4$~~

$$f(x)=(x-1)^2\ln\left(\sqrt{x^2-1}\right)$$

$$\Rightarrow f'(x)=2(x-1)\ln\left(\sqrt{x^2-1}\right)+(x-1)^2\frac{1}{\sqrt{x^2-1}}\cdot\frac{1}{\cancel{2}}(x^2-1)^{-1/2}\cdot\cancel{2}x=$$

$$=2(x-1)\ln\left(\sqrt{x^2-1}\right)+(x-1)^2\cdot\frac{1}{\sqrt{x^2-1}}\cdot\frac{x}{\sqrt{x^2-1}}=$$

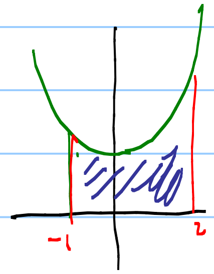
$$=2(x-1)\ln\left(\sqrt{x^2-1}\right)+\frac{x}{(x+1)}$$

Find the following areas:

Area problems:
(a) Area under $f(x)=x^2+1$ from $x=-1$ to $x=2$
(b) Area enclosed by $f(x)=4-x^2$ and x-axis
(c) Area under $f(x)=x^2-1$ from $x=0$ to $x=2$

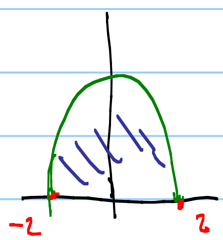
Solutions on next page

(a) area under $f(x)=x^2+1$, $x=-1$ & 2



f positive so $A=\int_{-1}^{2} x^2+1\,dx = \frac{1}{3}x^3+x\Big|_{-1}^{2} = \underline{\underline{6}}$

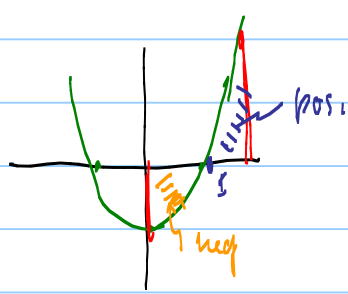
(b) area between $y=4-x^2$ and x-axis



f positive, bounds are $x=-2$ & $+2$

$A=\int_{-2}^{2} 4-x^2\,dx = \underline{\underline{32/3}}$

(c) area under $f(x)=x^2-1$ between $x=0$ and $x=3$



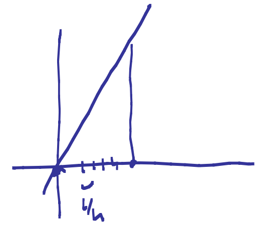
$\int_{0}^{1} x^2-1\,dx = \frac{1}{3}x^3-x\Big|_{0}^{1} = \frac{1}{3}-1 = -2/3$ neg

$\int_{1}^{2} x^2-1\,dx = \frac{1}{3}x^3-x\Big|_{1}^{2} = \left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right) =$

$= \frac{7}{3}-1 = \frac{4}{3}$ positive

$\Rightarrow$ Total area is $\frac{2}{3}+\frac{4}{3}=\frac{6}{3}=\underline{\underline{2}}$

Prove that $\int_0^1 2x\,dx = 1$, using the definition of the integral



$$\int_0^1 2x\,dx = \lim_{n\to\infty} f\left(\tfrac{1}{n}\right)\cdot\tfrac{1}{n} + f\left(\tfrac{2}{n}\right)\cdot\tfrac{1}{n} + \dots + f(1)\cdot\tfrac{1}{n} =$$

$$= \lim_{n\to\infty} \tfrac{1}{n}\left(2\cdot\tfrac{1}{n} + 2\cdot\tfrac{2}{n} + 2\cdot\tfrac{3}{n} + \dots + 2\tfrac{n}{n}\right) =$$

$$= \lim_{n\to\infty} \tfrac{2}{n^2}\left(1+2+\dots+n\right) = \lim_{n\to\infty} \frac{\cancel{2}}{n^2}\cdot\frac{n(n+1)}{\cancel{2}} = \lim_{n\to\infty}\frac{n(n+1)}{n^2} = 1$$

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Prove that *ln(x)* is increasing for *x > 0*

If $f(x) = \ln(x) = \int_1^x \frac{1}{t}\,dt$, then $f'(x) = \frac{1}{x}$, $x > 0$ by Fund Thm

But since $x > 0$ we have $f'(x) = 1/x > 0$ so f is increasing

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Prove that *ln(x b) = ln(x) + ln(b)*

Define $f(x) = \ln(xb) - \ln(x)$

$$\Rightarrow f'(x) = \frac{1}{x\cdot\cancel{b}}\cdot\cancel{b} - \frac{1}{x} = \frac{1}{x} - \frac{1}{x} = 0$$

That implies that $f(x)$ is constant, i.e.

$$\ln(xb) - \ln(x) = c$$

Let $x=1$: $\ln(b) - \ln(1) = c \Rightarrow c = \ln(b)$ because $\ln(1) = 0 \Rightarrow \ln(xb) - \ln(x) = \ln(b)$ or $\ln(xb) = \ln(x) + \ln(b)$

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Prove the MVT for integrals, using the Fund. Theorem of Calc and the ordinary MVT

Regular MVT: $\frac{g(b)-g(a)}{b-a} = g'(c)$, c in (a,b)

Let $F(x) = \int_a^x f(t)\,dt$. Then $F(a) = \int_a^a f(t)\,dt = 0$

(*) By regular MVT: $\frac{F(b)-F(a)}{b-a} = F'(c)$ for some c.

But $\frac{F(b)-F(a)}{b-a} = \frac{F(b)-0}{b-a} = \frac{1}{b-a} F(b) = \frac{1}{b-a} \int_a^b f(t)\,dt$

and by Fund. thm. of calculus

$$F'(c) = f(c)$$

Putting everything into equation (*) gives

$$\frac{1}{b-a} \int_a^b f(t)\,dt = f(c)$$

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