Math 1401: Practice Exam 2

Disclaimer: This is a *practice exam* only. It does not imply that any questions from this exam will indeed appear on the actual exam, or that questions on the actual exam will be taken exclusively from the questions appearing here. However, this practice exam should be a reasonably good approximation of the actual exam, except in the number of problems: this practice exam is longer than the actual exam.

Please find the derivative for each of the following functions (do not simplify unless it is helpful).

$fx) = \pi^2 + x^2 + \sin(x) + \sqrt{x}$ $\quad f'(x) = \underline{2x + \cos(x) + \frac{1}{2}x^{-1/2}}$

$f(x) = x^2(x^4 - 2x)$ $= x^6 - 2x^3 \rightarrow f'(x) = \underline{6x^5 - 6x^2}$

$f(x) = x^2\cos(x)$ $\quad f'(x) = \underline{2x\cos(x) - x^2\sin(x)}$

$f(x) = x\sin(x^2)$ $\quad f'(x) = \sin(x^2) + x\cos(x^2)\cdot 2x$

$f(x) = \dfrac{\sin(x)}{x^4 - 3}$ $\quad f'(x) = \dfrac{\cos(x)(x^4-3) - \sin(x)\cdot 4x^3}{(x^4-3)^2}$

$f(x) = \dfrac{\sec(x)}{x^4}$ $\quad f'(x) = \dfrac{\sec(x)\tan(x)4x^3 - \sec(x)4x^3}{x^8}$

$f(x) = \tan(x)\sqrt{x}$ $\quad f'(x) = \underline{\sec^2(x)\sqrt{x} + \tan(x)\frac{1}{2}x^{-1/2}}$

$f(x) = \pi^2\sin\left(\dfrac{\pi}{6}\right)$ $\quad \underline{f'(x) = 0}$

$f(x) = \dfrac{x^4 - 2x + 3}{x^2}$ $= x^2 - 2x^{-1} + 3x^{-2} \rightarrow f'(x) = \underline{2x + 2x^{-2} - 6x^{-3}}$

$f(x) = \dfrac{x^2}{x^2 - 1}$ $\quad f'(x) = \underline{\dfrac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2}}$

$f(x) = \dfrac{x\sin(x)}{x - 3}$

$f(x) = \dfrac{x^2\cos(x)}{(1-2x)^2}$ $\quad \underline{ok}$

$f(x) = x\sin(\sqrt{1 - x^2})$

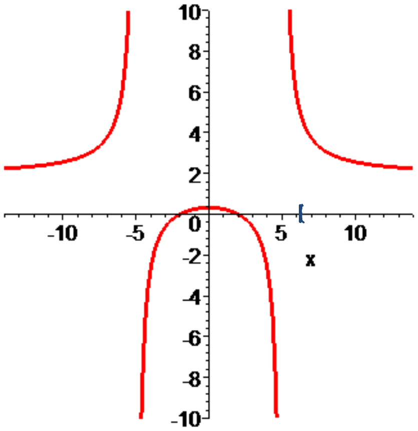
$f(x) = \sin^2(x) + \cos^2(x)$

$f(x) = \tan(x)$, find $f''(x)$

$f(x) = x\cos(x)$, find $f'''(x)$

$f(x) = 3x^5 - 2x^3 + 5x - 1$, find $f^{(7)}(x)$

For the function displayed below, find the following limits:



a) $\lim\limits_{x\to\infty} f(x) = 2$

b) $\lim\limits_{x\to-\infty} f(x) = -2$

c) $\lim\limits_{x\to5^+} f(x) = +\infty$

d) $\lim\limits_{x\to-5^+} f(x) = -\infty$

Also identify f' and f'' in a graph as discussed in class.

Suppose a function y is implicitly defined as a function of x via the equation $y^3 - 5x^2 = 3x$.

a) Find the derivative of y using implicit differentiation.

$3y^2y' - 10x = 3$

b) What is the equation of the tangent line at the point (1, 2).

$x=1, y=2 \Rightarrow 3\cdot4\cdot y' - 10 = 3$

$12y' = 13 \Rightarrow y' = \frac{13}{12}$

Find the slope of the tangent line to the graph of $y^4 + 3y - 4x^3 = 5x + 1$ at the point (1, -2), assuming that the equation defines *y* as a function of *x* implicitly.

$4y^3y' + 3y' - 12x^2 = 5$ at $x=1, y$

$x=1, y=-2,$ $4\cdot(-8)y' + 3y' - 12\cdot1 = 5$

$-29y' = 17 \Rightarrow y' = -\frac{17}{29}$

Find $\frac{dy}{dx}$ if $y = x^2\sin(y)$, assuming that *y* is an implicitly defined function of *x*.

etc

Suppose both x and y are both functions of t, implicitly defined via $x^2 + y^2 = xy$. Implicitly differentiate this equation with respect to t. Note that you do not have to solve for $\frac{dx}{dt}$ or $\frac{dy}{dt}$ in this problem.

$$2x\frac{dx}{dx} + 2y\frac{dy}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$$

or $\underline{2xx' + 2yy' = x'y + xy'}$

Find the following limits at infinity:

$$\lim_{x\to\infty}\frac{2x+3x^4}{4x^3-2x^2+x-1} = \infty$$

$$\lim_{x\to-\infty}\frac{x-x^5}{x^3-x^2+x-1} \sim \frac{-x^5}{x^3} = -x^2 \Rightarrow -\infty$$

$$\lim_{x\to-\infty}\frac{4x^3-2x^2+x-1}{2x-3x^4} = 0$$

$$\lim_{x\to-\infty}\frac{x^3-x^2+x-1}{x-3x^3} = -1/3$$

$$\lim_{x\to-\infty}\frac{\overbrace{(3x+4)(x-1)}^{x^2}}{\underbrace{(2x+7)(x+2)}_{x^2}} = 3/2$$

$$\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{x} = \lim\frac{\sqrt{x^2(1-\frac{1}{x^2})}}{x} = \lim_{x\to\infty}\frac{x\sqrt{1-\frac{1}{x^2}}}{\cancel{x}} = 1$$

Find all asymptotes, horizontal and vertical, if any, for the functions

$$f(x) = \frac{3x^2+1}{9-x^2}$$

$x = \pm 3$ vertical

$y = -3$ horizontal

$$f(x)=\frac{x^5}{1+x^4}$$ no asymptotes

$$f(x)=\frac{x-3}{x^2-5x+6} = \frac{(x-3)}{(x-3)(x-2)}$$ x=2 vertical asympt

y = 0 horizontal

If $f(x)=x^3+x^2-5x-5$, find the intervals on which f is increasing and decreasing, and find all relative extrema, if any.

$f'(x)=3x^2+2x-5=(3x+5)(x-1)=0 \Rightarrow x=-5/3,\ x=1$ are critical

	inc	$-5/3$ dec	1 inc
f'	+	−	+
f	↗	↘	↗

$x=-5/3$ is rel. max

$x=1$ is rel. min

$(-\infty,-5/3)\cup(1,\infty)$ incr.

$(-5/3,1)$ decreasing

Determine where the function $f(x)=x^4-2x^2$ is increasing and decreasing and find all relative extrema, if any.

same

Find the local maxima and minima for the function $f(x)=x^{\frac{1}{3}}(8-x)$

$$f'(x)=\frac{1}{3}x^{-2/3}(8-x)-x^{1/3}=\frac{8-x}{3x^{2/3}}-x^{1/3}=\frac{8-x}{3x^{2/3}}-x^{1/3}\cdot\frac{3x^{2/3}}{3x^{2/3}}=\frac{8-x-3x}{3x^{2/3}}=\frac{8-4x}{3x^{2/3}}$$

$\Rightarrow$ critical $x=0,2$ (0 because f' is undef at $x=0$)

		0	2
f'	+	+	−
f	↗	↗	↘

$x=2$ is local max

Find the absolute extrema (i.e. absolute maximum and absolute minimum) for the function $f(x) = 3x^4 - 6x^2$ on the interval [0, 2]

$f'(x) = 12x^3 - 12x = 12x(x^2-1) = 0 \Rightarrow x = 0, +1, -1$

abs. extrema:

	x	$f(x)$	
	0	0	
	1	-3	← abs. min
	~~-1~~		not in interval
end points	~~0~~		duplicate
	2	48 - 24 = 24	← abs max

Find the absolute maximum and minimum of the function $f(x) = 2x^3 + 3x^2 - 36x$ on the interval [0, 4].
Do the same for $f(x) = \frac{x}{x^2+1}$ on [0, 3], or for $f(x) = 3x^4 + 4x^3$ on [-2, 0].

$f(x) = 2x^3 + 3x^2 - 36x \Rightarrow f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x+3)(x-2) = 0$

critical $x = -3, +2$

	x	$f(x)$	
	~~-3~~		not in interval
	2	-44	← abs. min
end pts	0	0	
	4	32	← abs. max

If $f(x) = x^3 + x^2 - 5x - 5$, determine intervals on which the graph of f is concave up and intervals on which the graph is concave down.

$f'(x) = 3x^2 + 2x - 5$

$f''(x) = 6x + 2 = 0 \Rightarrow x = -\frac{1}{3}$

		$-1/3$
f''	$-$	$+$
f	$\cap$	$\cup$

$\Rightarrow$ Concave up: $(-1/3, \infty)$

concave down: $(-\infty, -1/3)$

If $f(x) = 12 + 2x^2 - x^4$, find all points of inflection and discuss the concavity of *f*. Do the same for $f(x) = x^5 - 5x^3$,

$f'(x) = 4x - 4x^3$

$f''(x) = 4 - 12x^2 = 0$

$x = \pm\sqrt{\frac{1}{3}}$

		$-1/\sqrt{3}$		$+\sqrt{1/3}$	
f''	$-$		$+$		$-$
f	$\cap$		$\cup$		$\cap$

cup: $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

down: $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$

Find the interval where $f(x) = 1 - x^{\frac{1}{3}}$ is concave up, if any.

ok

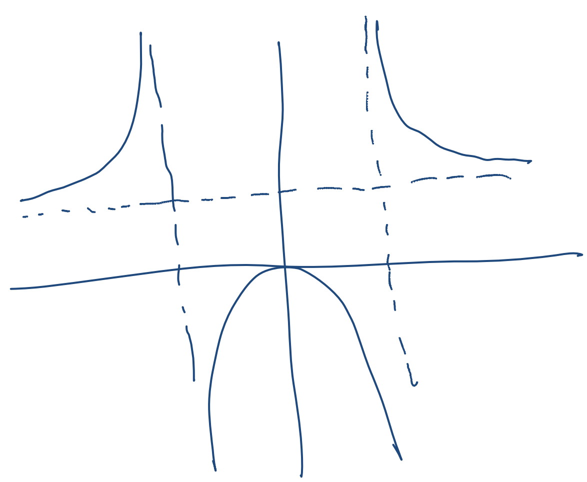
Graph the function $f(x) = \dfrac{x^2}{x^2 - 1}$. Note that $f'(x) = \dfrac{-2x}{(x^2-1)^2}$ and $f''(x) = \dfrac{2(3x^2+1)}{(x^2-1)^3}$.

horiz. asymp. $y = 1$

vert. asympt. $x = \pm 1$

critical: $x = 0, 1, -1$

poss. infl. $x = +1, -1$



		-1		0		1	
f'	$+$		$+$		$-$		$-$
f''	$+$		$-$		$-$		$+$
f							

$f(0) = 0$

Make sure to find all asymptotes (horizontal and vertical) and clearly label any maximum, minimum, and inflection points. Then do the same for the function $f(x)=\frac{8-4x}{(x-1)^2}$, or $\cancel{f(x)=\frac{2x^2-8}{x^2-16}}$, $\cancel{\text{or } f(x)=\frac{x^2-1}{x^3}}$

$f(x)=\frac{8-4x}{(x-1)^2}$, $f'(x)=\frac{4(x-3)}{(x-1)^3}$, $f''(x)=\frac{-8(x-4)}{(x-1)^4}$

vertical asy: $x=1$

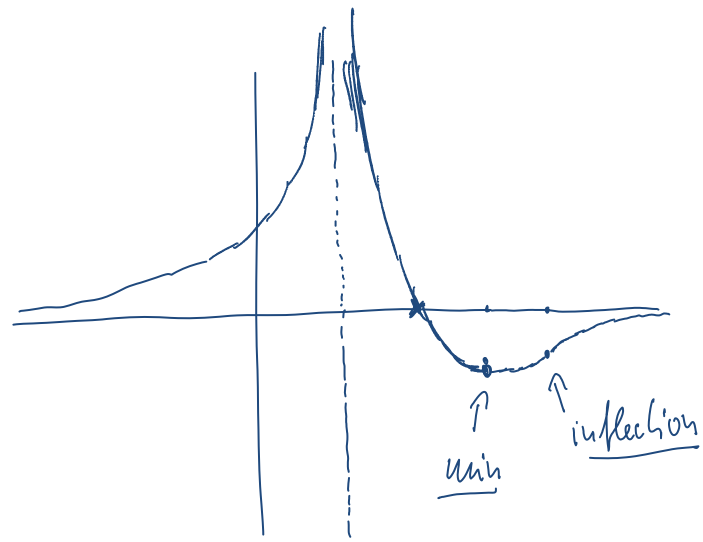
horizontal asy $y=0$

critical: $x=1, x=3$

poss. infl.: $x=4, 1$

$f(3)=-1$

$f(4)=-\frac{8}{9}$



		1	3	4
f'	+	−	+	+
f''	+	+	+	−
f				

A 13 meter ladder is leaning against a wall If the top of the ladder slips down the wall at a rate of 2 m/sec, how fast will the foot be moving away from the wall when the top is 5 m above the ground.

$x^2+y^2=13^2=169$

$x=x(t)$, $y=y(t)$

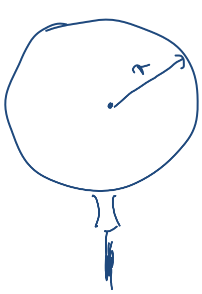
Know: $\frac{dy}{dt}=-2$

$y=5 \Rightarrow x=\sqrt{169-25}=12$

$2xx'+2yy'=0 \Rightarrow xx'=-yy' \Rightarrow x'=-y'\frac{y}{x}=2\cdot\frac{5}{12}=\underline{\underline{5/6}}$

Gas is escaping from a spherical balloon at a rate of 10 ft^3/hr. At what rate is the radius changing when the volume is 400 ft^3. $V=400$

$\hookrightarrow V'=-10$



$V=\frac{4}{3}\pi r^3$, $r=r(t)$

$V=\frac{4}{3}\pi r^3=400 \Rightarrow r^3=\frac{300}{\pi} \Rightarrow r\approx 9.77$

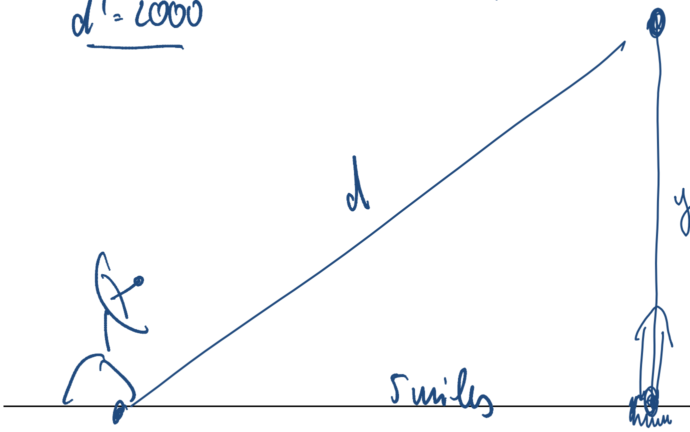
$\Rightarrow V'=4\pi r^2\cdot r'$

$\Rightarrow -10=4\pi\, 100\cdot r' \Rightarrow r'=\frac{-10}{400\pi}\approx\underline{\underline{-0.089}}$

A radar station that is on the ground 5 miles from the launch pad tracks a rocket, rising vertically. How fast is this rocket rising when it is 4 miles high and its distance from the radar station is increasing at a rate of 2000 mph ?

$y = 4$

$d' = 2000$



$d^2 = y^2 + 25$

$2dd' = 2yy' \Rightarrow dd' = yy'$

If $y = 4 \Rightarrow d^2 = 41 \Rightarrow d = \sqrt{41}$

$\Rightarrow y' = \frac{dd'}{y} = \frac{\sqrt{41}}{4} \cdot 2000$

A liquid form of penicillin manufactured by a pharmaceutical firm is sold in bulk at a price of \$200 per unit. If the total production cost (in dollars) for x units is C(x) = 500,000 + 80x + 0.003x^2 and if the production capacity of the firm is at most 30,000 units in a specified time, how many units of penicillin must be manufactured and sold in that time to ~~maximize~~ the ~~profit~~ ?

minimize cost

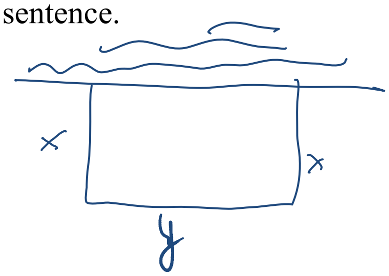
$C(x) = 500000 + 80x + 0.003x^2$, $x \in [0, 30000]$

$C'(x) = 80 + 0.006x = 0$

$\Rightarrow x = -13333.33$ not in interval

x	
0	← min
~~13333.33~~	
30000	← max

A farmer wants to fence in a piece of land that borders on one side on a river. She has 200m of fence available and wants to get a rectangular piece of fenced-in land. One side of the property needs no fence because of the river. Find the dimensions of the rectangle that yields maximum area. (Make sure you indicate the appropriate domain for the function you want to maximize). Please state your answer in a complete sentence.

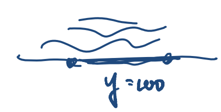


Know $2x + y = 200 \Rightarrow y = 200 - 2x$

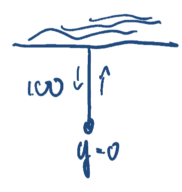
Max: $A = xy = x(200 - 2x) = 200x - 2x^2$

$x \in [0, 100]$

$x = 0$:



$x = 100$:



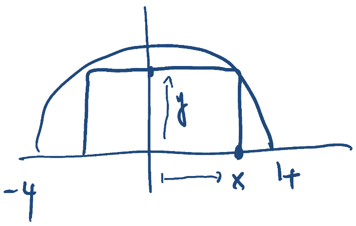
$A'(x) = 200 - 4x \Rightarrow x = 50$ is critical

x	A
0	0
100	0
50	2500 ← max

Use 50 m for x and 100 meter for y for max area

Find the dimensions of the rectangle of maximum area that can be inscribed in a semicircle of radius 16, if one vertex lies on the diameter.

$x^2+y^2=16$



$A = 2x \cdot y$

$x^2+y^2=16 \Rightarrow y = \sqrt{16-x^2} \Rightarrow A = 2x\sqrt{16-x^2}, \quad x \in [0,4]$

$\left.\begin{array}{l} \Rightarrow A'=0 \\ \text{Maple} \end{array}\right\} \Rightarrow x = \pm 2\sqrt{2}$

x	A
0	0
$2\sqrt{2}$	$\Leftarrow$ max
4	0

An open box with a rectangular base is to be constructed from a rectangular piece of cardboard 16 inches wide and 21 inches long by cutting out a square from each corner and then bending up the sides. Find the size of the corner square which will produce a box having the largest possible volume.

$V = (16-2x)(21-2x)\cdot x$

$x \in [0,8]$

Rest use Maple