Exam 2 –Practice

1. Please state what the following terms mean:
   a) Length of a curve, arc length
   b) Moment about x or y axis
   c) Center of Gravity
   d) What is an "infinite sequence"
   e) Increasing, decreasing, and bdd sequences + related theorems
   f) What is an "infinite series"
   g) What is the N-th partial sum
   h) The sequence $\{a_n\}$ converges to the limit L
   i) The series $\sum_{n=0}^{\infty} a_n$ converges to the limit L
   j) What is the Divergence Test?
   k) What is the Limit Comparison Test, and how about the Ratio Test
   l) What is a geometric series, a p-series
   m) What is the harmonic series?
   n) What is a Power Series?
   o) What is a Taylor Series? A McLaurin series?

2. Arc Length question: Find the length of the curve $y = 1 + 6x^{3/2}$ for 0 < x < 1

$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$. $f(x) = 1 + 6x^{3/2}$

$\Rightarrow f'(x) = 6 \cdot \frac{3}{2} x^{1/2} = 9x^{1/2}$

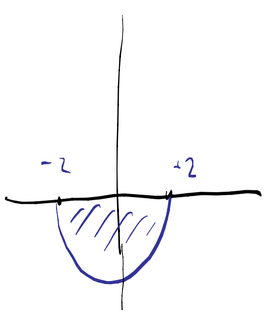
$\Rightarrow 1 + [f'(x)]^2 = 1 + 81x$

$$\Rightarrow L = \int_0^1 \sqrt{1+81x}\, dx = \int_0^1 (1+81x)^{1/2} dx = \frac{1}{81}\frac{2}{3}(1+81x)^{3/2}\Big|_0^1 = \frac{2}{3}\cdot\frac{1}{81}\left[82^{3/2} - 1\right]$$

3. Find the center of mass of the lamina of uniform density ρ bounded by the graph of $y = x^2 - 4$ and $y = 0$. You might need the following formulas:

$$M_x = \frac{\rho}{2}\int [f(x)]^2 dx \qquad M_y = \rho \int x \cdot f(x) dx \qquad m = \rho \int_a^b |f(x)| dx \qquad (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

$$m = \int_{-2}^{2} x^2 - 4\, dx = \frac{1}{3}x^3 - 4x\Big|_{-2}^{2} = \frac{8}{3} - 8 - \left(-\frac{8}{3} + 8\right) = \frac{16}{3} - 16 = -\frac{32}{3} \Rightarrow m = +\frac{32}{3}$$



$$M_y = \rho \int_{-2}^{2} x(x^2-4)\, dx = 0 \text{ because } x(x^2-4) \text{ is an odd function}$$

$$M_x = \rho \frac{1}{2} \int_{-2}^{2} (x^2-4)^2 dx = \frac{1}{2}\int_{-2}^{2} x^4 - 8x^2 + 16\, dx = \frac{1}{2}\left[\frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x \Big|_{-2}^{2}\right] =$$

$$= \frac{1}{2} 2 \cdot \left(\frac{32}{5} - \frac{64}{3} + 32\right) = \frac{96 - 320 + 480}{15} = \frac{256}{15}$$

Thus: $(\bar{x}, \bar{y}) = \left(0, -\frac{256}{15} \cdot \frac{3}{32}\right) = \left(0, -\frac{8}{5}\right)$

4. For the following sequences, please list the first 4 terms and determine what the limit of the sequence might be. If the sequence does not have a limit, please say so.

$\{(-1)^n\}_{n=1}^{\infty}$ = $-1, 1, -1, 1, -1, 1, \ldots$ diverges

$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ = $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converges to zero

$\left\{1 - \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ = $1+1, 1-\frac{1}{2}, 1+\frac{1}{3}, 1-\frac{1}{4}, 1+\frac{1}{5}, \ldots$ converges to $1 \pm 0 = 1$

$\left\{(-1)^n + \frac{4}{n}\right\}_{n=1}^{\infty}$ = $-1+4, +1+2, -1+\frac{4}{3}, +1+\frac{4}{4}, -1+\frac{4}{5}, \ldots \sim \pm 1 + 0$ diverges

$\left\{\frac{2^n}{n!}\right\}_{n=1}^{\infty}$ = $\frac{2}{1}, \frac{4}{2}, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \ldots$ converges to zero

$\{\cos(n)\}_{n=1}^{\infty}$ jumps around without pattern ⇒ diverges

$\left\{\frac{n-3}{2n^2+3n-6}\right\}_{n=10}^{\infty}$ $\sim \frac{n}{2n^2} \to 0$ converges to zero

$\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ $\left(1+\frac{1}{n}\right)^n$ converges to e:

Memorize

no necessary for exam 2

For Review:
Consider $\ln\left(1+\frac{1}{n}\right)^n = n \ln\left(1+\frac{1}{n}\right) = \frac{\ln(1+1/n)}{1/n}$

By L'Hospital: $\lim \frac{\ln(1+1/n)}{1/n} = \lim \frac{1}{1+\frac{1}{n}} = 1$ ⇒ original limit conv. to e^1

$\{a_n\}_{n=1}^{\infty}$ where $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$ (tricky)

$a_1 = 1, a_2 = \sqrt{7}, a_3 = \sqrt{6+\sqrt{7}}, a_4 = \sqrt{6+\sqrt{6+\sqrt{7}}}.$

Assume there was a limit: $\lim a_n = L = \lim a_{n+1} = \lim \sqrt{6+a_n} = \sqrt{6+L}$

$\Rightarrow L = \sqrt{6+L} \Rightarrow L^2 = 6+L \Rightarrow L^2 - L - 6 = 0 \Rightarrow (L-3)(L+2) = 0 \Rightarrow L = 3$ ~~$L=-2$~~

not likely on the exam

It remains to prove that there is a limit.

① $a_n \le 3$ by induction.

$n=1$: $a_1 = 1 < 3$ ✓

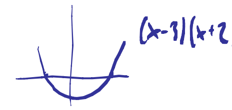
Assume $a_n \le 3$

$a_{n+1} = \sqrt{6+a_n} \le \sqrt{6+3} = \sqrt{9} = 3$

$\Rightarrow a_{n+1} \le 3$

② a_n is increasing because

know $0 \le a_n \le 3$ from ①



$\Rightarrow (a_n - 3)(a_n + 2) \le 0$

$\Rightarrow a_n^2 - a_n - 6 \le 0$

$\Rightarrow a_n^2 \le 6 + a_n$

$\Rightarrow a_n \le \sqrt{6+a_n}$

$\Rightarrow a_n \le a_{n+1} \Rightarrow a_n$ increasing

5. Questions about inc/dec/bdd sequences TBD

Now $\{a_n\}$ is increasing + bounded $\Rightarrow$ must converge.

If it converges, the limit must be 3.

6. Determine whether each of the following series converge (absolutely) or diverge. Please state carefully which test you are using to support your conclusion. If possible, find the limit of the series

$\sum_{n=1}^{\infty} \frac{n}{\ln(n)}$ $\quad \lim \frac{n}{\ln(n)} = \infty$ so series diverges by divergence test!

$\sum_{n=1}^{\infty} \underbrace{\frac{n-1}{n^3+n+1}}_{a_n}$ $\quad$ Consider $\sum \underbrace{\frac{1}{n^2}}_{b_n}$: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{n-1}{n^3+n+1} \cdot \frac{n^2}{1} = \lim \frac{(n-1)n^2}{n^3+n+1} = 1$

Thus by limit comp. test the series compares to $\sum 1/n^2$ (p=2-series) so that both series converge.

$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (n+1) \cdot 3^n}{n!}$

Ratio Test: $\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{(n+2)3^{n+1}}{(n+1)!} \cdot \frac{n!}{(n+1)3^n} \right| = \lim_{n\to\infty} \frac{3(n+2)}{(n+1)^2} = 0 < 1$

series converges

$\sum_{n=5}^{\infty} \frac{3^n}{5^n}$ $\quad$ Geometric series with $r = 3/5 \Rightarrow$ converges. Can find actual limit:

$\sum_{n=5}^{\infty} \left(\frac{3}{5}\right)^n = \left(\frac{3}{5}\right)^5 + \left(\frac{3}{5}\right)^6 + \left(\frac{3}{5}\right)^7 + \dots =$

$= \left(\frac{3}{5}\right)^5 \left[1 + \left(\frac{3}{5}\right) + \left(\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^3 + \dots\right] =$

$= \left(\frac{3}{5}\right)^5 \frac{1}{1-3/5} = \left(\frac{3}{5}\right)^5 \cdot \frac{1}{2/5} = \left(\frac{3}{5}\right)^5 \cdot \frac{5}{2}$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^n}{3^n}$$

Geometric series with $r = -2/3$. Converges to

$$\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n = \frac{1}{1+2/3} = \frac{1}{5/3} = \underline{\underline{3/5}}$$

$$\sum_{n=2}^{\infty}\frac{4}{n(n-1)}$$

Converges by limit comp. with $\sum 1/n^2$. But is also a telescoping series:

$\frac{4}{n(n-1)} = \frac{4}{n-1} - \frac{4}{n}$ by PF Decomp. Therefore

$$\sum_{n=2}^{\infty}\frac{4}{n(n-1)} = \sum_{n=2}^{\infty}\frac{4}{n-1}-\frac{4}{n} = \left(\frac{4}{1}-\frac{4}{2}\right)+\left(\frac{4}{2}-\frac{4}{3}\right)+\left(\frac{4}{3}-\frac{4}{4}\right)+\left(\frac{4}{4}-\frac{4}{5}\right)+\dots =$$

$$= \underline{\underline{4}}$$

7. Find the radius of converges for the following power series

$$\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$$

ratio test: $\lim\left|\frac{x^{n+1}}{4^{n+1}}\cdot\frac{4^n}{x^n}\right| = \frac{1}{4}|x| < 1 \Rightarrow |x| < 4 \Rightarrow$

radius of convergence is $\underline{\underline{R=4}}$

$$\sum(-1)^{n+1}\frac{x^{2n+1}}{n\cdot 2^{2n+1}}$$

$$\lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)2^{2n+3}}\cdot\frac{n2^{2n+1}}{x^{2n+1}}\right| = \lim\left|\frac{x^2\cdot x^{2n+1}\cdot n\, 2^{2n+1}}{(n+1)2^2\cdot 2^{2n+1}\cdot x^{2n+1}}\right| =$$

$$= \lim\frac{n}{n+1}\cdot\frac{1}{4}|x|^2 = \frac{1}{4}|x|^2 < 1$$

$$\Rightarrow |x|^2 < 4 \Rightarrow |x| < 2 \Rightarrow \underline{\underline{R=2}}$$

$$\sum_{n=1}^{\infty}\frac{n^n x^n}{n!}$$

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n x^n}\right| = \lim\frac{(n+1)^n\cdot(n+1)}{(n+1)\,n^n}|x| =$$

$$= \lim\left(\frac{n+1}{n}\right)^n|x| = \lim\left(1+\frac{1}{n}\right)^n\cdot|x| =$$

$$= e|x| < 1 \Rightarrow |x| < 1/e \Rightarrow \underline{\underline{R = 1/e}}$$

$$\frac{1}{1-[\ ]} = \sum [\ ]^n$$

8. Recall that $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x}$ for $|x| < 1$. Use that fact to determine the power series centered at the origin for:

a) $f(x) = \frac{1}{1-4x^2}$

$$= \frac{1}{1-(4x^2)} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n}$$

b) $g(x) = -\ln(1-x)$

$$-\ln(1-x) = -\int \frac{1}{1-x}\,dx = -\int \sum_{n=0}^{\infty} x^n\,dx = -\sum_{n=0}^{\infty} \int x^n\,dx =$$

$$= -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

c) $h(x) = \frac{x^5}{1+x}$

$$= \frac{x^5}{1-(-x)} = x^5 \cdot \frac{1}{1-(-x)} = x^5 \sum_{n=0}^{\infty} (-x)^n = x^5 \sum_{n=0}^{\infty} (-1)^n x^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{n+5}$$

9. Find the Taylor series for the following functions, all to be centered at the origin.

a) $x^3 e^{x^2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\Rightarrow x^3 e^{x^2} = x^3 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+3}}{n!}$$

b) $\frac{\cos(x)-1}{x^2}$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\Rightarrow \cos(x) - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\frac{\cos(x)-1}{x^2} = x^{-2}\left(\cos(x)-1\right) = -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \ldots = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-2}}{(2n)!}$$

c) $\int e^{-x^2}\,dx$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$\Rightarrow \int e^{-x^2}dx = \int 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{2n}}{n!} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\, n!} + C$$

d) $e^x \cos(x)$ (first 3 terms only)

Alternative: find $f(0), f'(0), f''(0), f'''(0), \dots$ $\Rightarrow a_0 = f(0),\ a_1 = \frac{f'(0)}{1!},\ a_2 = \frac{f''(0)}{2!},\ a_3 = \frac{f'''(0)}{3!}$ etc.

$$e^x \cos(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) =$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \text{higher} + x - \frac{x^3}{2!} + \frac{x^5}{4!} - \text{higher} + \frac{x^2}{2!} - \frac{x^4}{2!2!} + \frac{x^6}{2!4!} - \text{higher} + \frac{x^3}{3!} + \text{higher} + \frac{x^4}{4!} + \text{higher}$$

$$= 1 + x + 0x^2 + x^3\left(\frac{1}{3!} - \frac{1}{2!}\right) + x^4\left(\frac{2}{4!} - \frac{1}{2!2!}\right) + \text{higher} \Rightarrow a_0 = 1, a_1 = 1, a_2 = 0, a_3 = -\tfrac{1}{3}, a_4 = -\tfrac{1}{6}, \dots$$

10. Suppose the indicated function has a power series *around 0*. Find the value of the specified term:

a) $f(x) = \sin(2x)\cos(3x)$, find a_1 $\quad a_1 = \frac{f'(0)}{1!} = f'(0)$ so we need $f'(x)$, then $f'(0)$.

$f'(x) = 2\cos(2x)\cos(3x) - 3\sin(3x)\sin(2x)$

$\Rightarrow f'(0) = 2 \Rightarrow a_1 = 2$

b) $f(x) = \tan(x)$, find a_2 $\quad a_2 = \frac{f''(0)}{2!}$ so need f', f'', then $f''(0)$.

$f(x) = \tan(x)$

$f'(x) = \sec^2(x)$

$f''(x) = 2\sec(x) \cdot \sec(x)\tan(x) = 2\sec^2(x)\tan(x)$

$\Rightarrow f''(0) = 0 \Rightarrow a_2 = 0$

a) $f(x) = \sqrt{2+x}$, find a_5 $\quad a_5 = \frac{f^{(5)}(0)}{5!}$

$\Rightarrow f(x) = (2+x)^{1/2}$

$f'(x) = \frac{1}{2}(2+x)^{-1/2}$

$f''(x) = -\frac{1}{4}(2+x)^{-3/2}$

$f'''(x) = \frac{3}{8}(2+x)^{-5/2}$

$f^{(4)}(x) = -\frac{15}{16}(2+x)^{-7/2}$

$f^{(5)}(x) = \frac{105}{32}(2+x)^{-9/2} \Rightarrow f^{(5)}(0) = \frac{105}{32} \cdot \frac{1}{2^{9/2}}$

$$a_5 = \frac{105}{32} \cdot \frac{1}{2^{9/2}} \cdot \frac{1}{5!} = \frac{7}{8192} \cdot \sqrt{2}$$

11. The series $\sum_{n=1}^{\infty}\left(\frac{9}{10}\right)^n$ converges (why).

Geometric series with $r = 9/10 < 1$ converges

What number does it converge to?

$$\sum_{n=1}^{\infty}\left(\frac{9}{10}\right)^n = \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^3 + \dots = \frac{9}{10}\left(1 + \left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^3 \dots\right) = \frac{9}{10}\cdot\frac{1}{1-9/10} = \frac{9}{10}\cdot\frac{10}{1} = 9$$

What about $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^{n-1}$ Tricky: Know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum x^n$$

$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}\frac{d}{dx}x^n = \sum_{n=1}^{\infty} n x^{n-1}$$

Thus

$$\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^{n-1} = \frac{1}{(1-2/3)^2} = 9$$

12. Find a Taylor series for the function $\arctan(x)$. Tricky

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \Rightarrow \arctan(x) = \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1-(-x^2)}\,dx = \int \sum (-1)^n x^{2n}\,dx =$$

$$= \sum \int (-1)^n x^{2n}\,dx =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

Use that series together with the fact that $\arctan(1) = \frac{\pi}{4}$ to find a series that converges to π.

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1} \Rightarrow \frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}(1)^{2n+1}$$

$$\Rightarrow \pi = 4\cdot\left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\right)$$

Finally, use the first 5 terms of that series to get an approximate value for π .

$$\pi = 4 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots\right] \approx 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right) =$$

$$= 4 \cdot \frac{263}{315} = \underline{\underline{3.3398}}$$

<u>Note:</u> This is the first time you have seen something that is easy to compute yet converges to the complicated number π.

<u>Note:</u> We could now define both e and π as series:

$$\underline{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = \underline{\underline{\sum_{n=0}^{\infty} \frac{1}{n!}}}$$

$$\underline{\underline{\pi}} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \underline{\underline{\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}}}$$