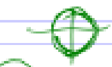
Panel 1

Last Time



$(x(t), y(t))$

Parametric curves: $(x,y) = (f(t), g(t))$ $\quad x = t^2 \Rightarrow t = \sqrt{x}$

Example: ① $(x,y) = (t^2, t^3 - 2t^4)$ $\quad y = (\,)^3 - 2(\,)^4 = (x)^{3/2} - 2x^2$

$(x,y) = (\cos(t), \sin(t))$ $\quad x^2 + y^2 = 1$

$x' = 2t = 0,\ y' = 3t^2 - 8t^3$

$t = 0$

Smooth curves: $x'(t)$ and $y'(t)$ not simultaneously zero

no kinks

slope of Tangent to curve: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{y'}{x'}$

1

Panel 2

Find slope of tangent to $\left(\sqrt{t}, \frac{1}{4}(t^2-4)\right)$ at $(2, 3)$

① $x = \sqrt{t}$, $\Rightarrow t = x^2$ $\qquad$ ② $\frac{1}{4}(t^2-4) \Rightarrow t = 4$

$y = \frac{1}{4}(t^2-4) \Rightarrow y = \frac{1}{4}(x^4-4)$

$y' = \frac{1}{4} \cdot 4x^3 = x^3 \big|_{x=2} \Rightarrow y'$ at $x=2$ is $\underline{8}$

② $x'(t) = \frac{1}{2} t^{-1/2}$ at $t = 4$: $x'(t) = \frac{1}{4}$

$y'(t) = \frac{1}{2}t$ at $t = 4$: $y'(t) = 2$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{2}{1/4} = \underline{8}$

2

Panel 3

Ex: Consider $(t^2, t^3 - 3t)$

a) Is it smooth?

$x'(t) = 2t = 0 \Rightarrow t = 0$

$y'(t) = 3t^2 - 3$ $\quad$ not zero at $t = 0$

So smooth

no kinks

3

Panel 4

Ex: Consider $(t^2, t^3 - 3t)$

b) Has two tangents at $(3, 0)$ - find them.

$t^2 = 3$ and $t^3 - 3t = 0$

$t = \pm\sqrt{3}$ $\quad 3\sqrt{3} - 3\sqrt{3} = 0$, $\quad$ both t's give that point

$-3\sqrt{3} + 3\sqrt{3} = 0$ $\quad (3, 0)$

$\dfrac{y'}{x'} = \dfrac{3t^2 - 3}{2t}$ at $t = \sqrt{3}$: $\dfrac{6}{2\sqrt{3}} \Rightarrow y - 0 = \dfrac{6}{2\sqrt{3}}(x - 3)$

at $t = -\sqrt{3}$: $\dfrac{6}{-2\sqrt{3}} \Rightarrow y - 0 = \dfrac{-6}{2\sqrt{3}}(x - 3)$

4

Panel 5

Ex: (t^2, t^3-3t) - find horizontal / vertical tangents.

$\frac{dy}{dx} = \frac{3t^2-3}{2t}$ is slope of tangent.

horizontal: slope = zero: $\frac{3t^2-3}{2t} = 0 \iff 3t^2-3=0 \iff t=\pm 1$

vertical: slope $= \pm\infty$: $\frac{3t^2-3}{2t} = \pm\infty \iff t=0$

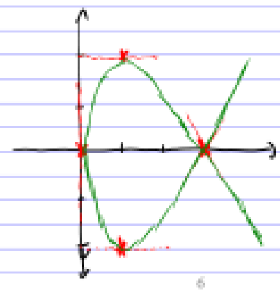
Panel 6

Ex: (t^2, t^3-3t) - sketch curve

Know: two tangents at $(3,0)$ with pos./neg. slope

horiz. tangent at $t=\pm 1$: $(1,-2)$ and $(1,+2)$

vertical tangent at $t=0$: $(0,0)$



Panel 7

Ex: Find the arc length of $(x(t), y(t))$, $t \in [a,b]$

Recall: $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx =$

$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\frac{dx}{dt}\,dt =$

$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2\left(1+\left(\frac{dy}{dx}\right)^2\right)}\,dt =$

$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{(x')^2+(y')^2}\,dt$

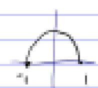
Formula

$\Downarrow$

Panel 8

Ex: Find the length of the unit circle.

① $y = \sqrt{1-x^2}$: $L = 2\int_{-1}^{1} \sqrt{1+(y')^2}\,dx$ too much work



② $(\cos(t), \sin(t))$, $t \in [0, 2\pi]$ $\Rightarrow$

$L = \int_0^{2\pi} \sqrt{(x')^2+(y')^2}\,dt = \int_0^{2\pi} \sqrt{\sin^2(t)+\cos^2(t)}\,dt =$

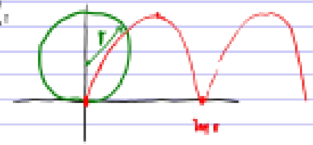
$= \int_0^{2\pi} dt = 2\pi$

Panel 9

see book

Find length of $(r(t - \sin(t)), r(1 - \cos(t)))$, $t \in [0, 2\pi]$

cycloid:



roll green wheel to positive x-axis, red curve is cycloid

$x'(t) = r(1 - \cos(t))$, $y'(t) = r\sin(t)$ not smooth at $t = 0, 2\pi, 4\pi, \ldots$

$$L = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\, dt = \int_0^{2\pi} \sqrt{r^2(1-\cos(t))^2 + r^2 \sin^2(t)}\, dt =$$

$$= r \int_0^{2\pi} \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}\, dt = r \int_0^{2\pi} \sqrt{2 - 2\cos(t)}\, dt$$

9

Panel 10

Need: $L = r \int_0^{2\pi} \sqrt{2 - 2\cos(t)}\, dt = ?\ 8r$

why? HW

10