Practice Exam 3

Part 2 of the Answers

Find the derivatives of the following functions. You might use logarithmic differentiation if that simplifies your task:

$f(x) = 5^x + \log_2(1 + x^2)$

$f'(x) = \ln(5)\, 5^x + \frac{1}{\ln(2)} \frac{2x}{1+x^2}$

Find the following limits. You might want to use l'Hospital's rule where appropriate

$\lim_{x\to\infty} x^3 e^{-x}$

$\lim_{x\to\infty} \frac{x^3}{e^x} = \lim_{x\to\infty} \frac{3x^2}{e^x} = \lim_{x\to\infty} \frac{6x}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0$

$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}}$

$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{\mathrm{x}^{-1}}{\frac{1}{3}x^{-\frac{2}{3}}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0$

$\lim_{x\to 0} \frac{e^{3x}-3t-1}{x^2-x}$

$\lim_{x\to 0} \frac{e^{3x}-3t-1}{x^2-x} = \lim_{x\to 0} \frac{3e^{3x}-3}{2x-1} = 0$

The edge of a cube was found to be 30 cm with a possible error of 0.1 cm. Use differentials to estimate the maximum possible error and the relative error in computing (a) the volume of the cube and (b) the surface area of the cube.

$V = x^3$ so that $dA = 3x^2\, dx = 3 * 30^2 * 0.1 = 270$. Since $A = 30^3$ the relative error is $\frac{dA}{A} = \frac{270}{27000} = 0.01 = 1\%$

$S = 6x^2$ so that $dS = 12x dx = 12 * 30 * 0.1 = 36$. Since $S = 6 * 30^2 = 5400$ the reative error is $\frac{dS}{S} = \frac{36}{5400} = 0.00666 = 0.67\%$

15. Find the antiderivatives of the functions

a) $f(x) = 3x^2 + \frac{1}{x} + e^x - \sin(x)$

$F(x) = x^3 + \ln(x) = cos(x)$

b) $g(x) = \frac{3}{1+x^2}$

$G(x) = 3\tan^{-1}(x)$

16. Which of the following functions is the antiderivative of $f(x) = \ln(x)$?
    a) $F(x) = \frac{1}{x}$ no
    b) $F(x) = \frac{1}{2}(\ln(x))^2$ no
    c) $F(x) = xln(x) - x$ $\quad F' = 1\ln(x) + x\frac{1}{x} - 1 = \ln(x)$ yes

17. Solve the following initial value problem:
    a) Find $f(x)$ such that $f'(x) = 4x^5 - 3x^2$ and $f(1) = 2$

$f(x) = \frac{4}{6}x^6 - x^3 + C$ and since $f(1) = \frac{4}{6} - 1 + C = 2$ we have that $C = 3 - \frac{4}{6} = \frac{14}{6} = \frac{7}{3}$

b) Find $f(x)$ such that $f''(x) = 12x^2 - 6x$ and $f'(0) = 2$ and $f(0) = 2$

$f'(x) = 4x^3 - 3x^2 + C$ and since $f'(0) = C = 2$ we have $f'(x) = 4x^3 - 3x^2 + 2$ But then $f(x) = x^4 - x^3 + 2x + D$ and again D = 2. So the answer is: $f(x) = x^4 - x^3 + 2x + 2$

18. You are standing on the rim of deep hole in the ground, that is so deep you cannot see the bottom. You drop a stone into the hole and you notice that it hits the ground after 10 seconds. How deep is the hole?

$a = -g = -32$ so that $v(t) = -32t + v_0$ and $s(t) = -16t^2 + v_0 t + s_0$
The initial velocity $v_0 = 0$ and the initial height is $s_0 = 0$ as well. Thus, after 10 seconds we have $s(10) = -16 * 10^2 = -1600$ feet, which would be the depth of the hole.

19. Bert is at bat and he happens to hit the incoming baseball 4 feet above the ground so that it goes straight up with initial velocity of 50 ft/sec. There is a runner on third base and it would take him 10 seconds to run home. Will he make it before the ball hits the ground?

Again $a = -g = -32$ so that $v(t) = -32t + v_0$ and $s(t) = -16t^2 + v_0 t + s_0$
The initial velocity $v_0 = 50$ and the initial height is $s_0 = 4$. Thus, $s(t) = -16t^2 + 50t + 4$. Now the time for the ball to hit the ground would be $s(t) = 0$ or $-16t^2 + 50t + 4 = 0$. Thus, $t = -0.07805$, $t = 3.20305$. Time cannot be negative, so the ball hits the ground after 3.2 seconds, which is **not** enough for the runner on third to score.

20. Extra credit: Prove that the derivative of the *inverse sine* function is $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ and that of the *inverse tangent* is $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$