45–54 ▪ Use logarithmic differentiation or an alternative method to find the derivative of the function.

45. $y = (2x + 1)^5(x^4 - 3)^6$

46. $y = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}$

47. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

48. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$

49. $y = x^x$

50. $y = x^{\cos x}$

51. $y = (\cos x)^x$

52. $y = \sqrt{x}^{\,x}$

53. $y = (\tan x)^{1/x}$

54. $y = (\sin x)^{\ln x}$

55. Find y' if $e^{x^2 y} = x + y$.

56. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

57. Find y' if $y = \ln(x^2 + y^2)$.

58. Find y' if $x^y = y^x$.

59. The motion of a spring that is subject to a frictional force or a damping force (such as a shock absorber in a car) is often modeled by the product of an exponential function and a sine or cosine function. Suppose the equation of motion of a point on such a spring is

$$s(t) = 2e^{-1.5t} \sin 2\pi t$$

where s is measured in centimeters and t in seconds. Find the velocity after t seconds and graph both the position and velocity functions for $0 \leq t \leq 2$.

60. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants.

(a) Find $\lim_{t \to \infty} p(t)$.

(b) Find the rate of spread of the rumor.

(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

61. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

62. For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 5y' - 6y = 0$?

63. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

64. Find the thousandth derivative of $f(x) = xe^{-x}$.

65. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

66. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

67. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

68. Evaluate $\displaystyle\lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi}$.

3.4 EXPONENTIAL GROWTH AND DECAY

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant k. Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what actually happens fairly accurately. Another example occurs in nuclear physics where the mass of a radioactive substance decays at a rate proportional to the mass. In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance. In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

1
$$\frac{dy}{dt} = ky$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth**