Practice Exam 3

Partial – additional problems are posted tomorrow

1. State the definition or meaning of the following terms:

   a) What is the definition of the inverse function to a given function *f(x)*? If *f* is differentiable, what is the relation between the derivative of the inverse function and the derivative of the function?

   b) What is the definition of *ln(x)*, $sin^{-1}(x)$, $cos^{-1}(x)$, and $tan^{-1}(x)$. What is their derivative?

   c) What is l'Hopital's Rule and why is it so useful?

   d) What is "logarithmic differentiation" and when is it helpful?

   e) What is "exponential growth" and "exponential decay"?

2. Find the derivatives of the following functions. You might use logarithmic differentiation if that simplifies your task:

$f(x) = 2e^{5x^2} + 7\ln(x^2+3)$

$f'(x) = 2e^{5x^2} \cdot 10x + \frac{7 \cdot 2x}{x^2+3}$

$f(x) = \ln\left(\frac{\sqrt{x-1}}{(x-1)^2}\right) = \frac{1}{2}\ln(x-1) - 2\ln(x-1)$

$f'(x) = \frac{1}{2(x-1)} - \frac{2}{x-1}$

$f(x) = (x-1)^2 \sin^{-1}(x^2)$

$f'(x) = 2(x-1)\sin^{-1}(x^2) + (x-1)^2 \cdot \frac{2x}{\sqrt{1-x^4}}$

$f(x) = \sin^{-1}(1-x) + \cos^{-1}(1-x)$

$f'(x) = \frac{1}{\sqrt{1-(1-x)^2}} \cdot (-1) - \frac{1}{\sqrt{1-(1-x)^2}}(-1) = 0$

$f(x) = \frac{\tan^{-1}(x)}{x}$

$f'(x) = \frac{\frac{1}{1+x^2}x - \tan^{-1}(x) \cdot 1}{x^2}$

$f(x) = \frac{\cos^{-1}(4x)}{x\ln(3x)}$

$f' = \frac{\frac{-1}{\sqrt{1-16x^2}} \cdot x\ln(3x) - \cos^{-1}(4x)\left[\ln(3x) + \cancel{x}\frac{1}{3\cancel{x}} \cdot 3\right]}{(x\ln(x))^2}$

$f(x) = x^2 \sin^{-1}(3x)$

$f' = 2x \sin^{-1}(x) + x^2 \frac{3}{\sqrt{1-9x^2}}$

$g(x) = \frac{\arctan(3x)}{\arccos(2x)}$

$g'(x) = \frac{\frac{3}{1+9x^2} \cdot \sin^{-1}(2x) + \tan^{-1}(3x) \frac{1}{\sqrt{1-4x^2}} \cdot 2}{(\cos^{-1}(x))^2}$

$h(x) = x\sqrt{\tan^{-1}(x^2)}$

$h'(x) = \sqrt{\tan^{-1}(x^2)} + x \cdot \frac{1}{2} (\tan^{-1}(x^4))^{-1/2} \cdot \frac{1}{1+x^4} \cdot 2x$

$y = f(x) = \frac{(x-1)^2}{(x+1)^3}(x+2)^4$

$\ln(y) = 2\ln(x-1) + 4\ln(x+2) - 3\ln(x+1) \Rightarrow \frac{y'}{y} = \frac{2}{x-1} + \frac{4}{x+2} - \frac{3}{x+1}$

$\Rightarrow y' = \left(\frac{2}{x-1} + \frac{4}{x+2} - \frac{3}{x+1}\right) y$

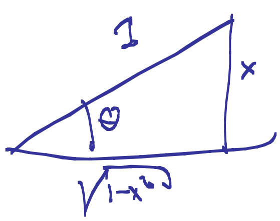
$y = g(x) = \frac{\sqrt{x^2-1}}{x^5(x-4)^4}$ $\ln(y) = \frac{1}{2}\ln(x^2-1) - 5\ln(x) - 4\ln(x-4) \Rightarrow \frac{y'}{y} = \frac{1}{2(x^2-1)} - \frac{5}{x} - \frac{4}{x-4}$

3. Simplify the following expressions

$ln(\frac{x^3y}{z^4}) = 3\ln(x) + \ln(y) - 4\ln(z)$

$2\log_2(4) - \log_2(2) = \log_2\left(\frac{4^2}{2}\right) = \log_2(8) = 3$

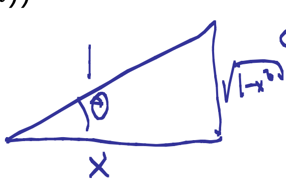
$\tan(\sin^{-1}(x))$



$\sin^{-1}(x) = \theta$

$\Rightarrow \frac{x}{1} = \sin(\theta) = opp/hyp \Rightarrow \tan(\theta) = \frac{opp}{adj} = \frac{x}{\sqrt{1-x^2}}$

$\sin(\cos^{-1}(x))$



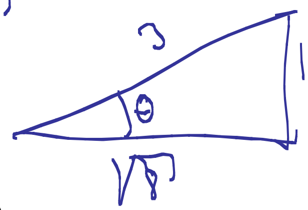
$\cos^{-1}(x) = \theta$

$\frac{x}{1} = \cos(\theta) = \frac{adj}{hyp} \Rightarrow \sin(\theta) = \frac{opp}{hyp} = \sqrt{1-x^2}$

$\sin^{-1}(\frac{1}{2}) = y$

$\Rightarrow \frac{1}{2} = \sin(y) \Rightarrow y = \pi/6$

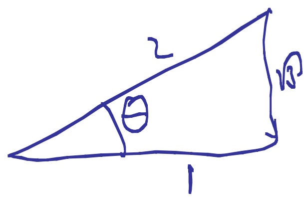
tan (arcsin$\left(\frac{1}{3}\right)$)



$\sin^{-1}\left(\frac{1}{3}\right) = \theta$

$\frac{1}{3} = \sin(\theta) = {}^{opp}/_{hyp}$ $\Rightarrow \tan(\theta) = {}^{opp}/_{adj} = \underline{\underline{1/\sqrt{8}}}$

cos (sin^{-1}($\frac{\sqrt{3}}{2}$))



$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \theta$

$\frac{\sqrt{3}}{2} = \sin(\theta) \Rightarrow \cos(\theta) = \underline{\underline{1/2}}$

4. The half-life or radium-226 is 1590 years. A sample of radium-226 has a mass of 100 mg. Find a formula about how much of the substance remains after t years. Find the mass after 1000 years. Also, find out how long it takes until the original mass of 100mg is reduced to 80 mg.

Model $A = A_0 e^{kt} = 100e^{kt}$

$\Rightarrow A(1590) = 100e^{k\cdot 1590} = 50$ (1/2 the material)

$\Rightarrow e^{k\cdot 1590} = 1/2 \Rightarrow k\cdot 1590 = \ln(1/2) = \ln(1) - \ln(2) = -\ln(2)$

$\Rightarrow k = \frac{-\ln(2)}{1590} = \underline{-0.000436}$

$\Rightarrow$ After 1000 years:

$A = 100e^{-0.000436\cdot 1000} = \underline{\underline{64.66}}$

$80 = 100e^{-0.000436\cdot t} \Rightarrow$

$0.8 = e^{-0.000436t} \Rightarrow t = \frac{\ln(0.8)}{-0.000436}$

$= \underline{\underline{511 \text{ years}}}$

Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960. Assuming exponential growth, what would the population of the world be in 2020?

$t=0$ means 1950. $P = 2560e^{kt}$

$P(10) = 2560e^{10k} = 3040 \Rightarrow k = \ln\left(\frac{3040}{2560}\right)\cdot\frac{1}{10} = \underline{\underline{0.0172}}$

$\Rightarrow$ in 2020: $P(70) = 2560e^{0.0172\cdot 70} = \underline{\underline{8524 \text{ Mill}}}$

5. Find the following limits. You might want to use l'Hospital's rule (but not all limits require that and for some it might not be appropriate)

$\lim_{x\to\infty} \arctan(x) = \pi/2$ (from graph)

$\lim_{x\to\infty} e^{-x^2} = 0$ (from graph)

$\lim_{x\to 3} \frac{x^2-9}{x+3} = \frac{0}{6} = 0$

$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \frac{0}{0} \overset{\text{l'Hop}}{=} \lim_{x\to 1} \frac{1/x}{1} = 1$

$\lim_{x\to 3} \frac{x^2-9}{x-3} = \frac{0}{0} \overset{\text{l'Hop}}{=} \lim_{x\to 3} \frac{2x}{1} = 6$

$\lim_{x\to \infty} \frac{x^2}{e^x} = \frac{\infty}{\infty} \overset{\text{l'Hop}}{=} \lim_{x\to\infty} \frac{2x}{e^x} = \frac{\infty}{\infty} \overset{\text{l'Hop again}}{=} \lim \frac{2}{e^x} = \frac{2}{\infty} = 0$

$\lim_{x\to 0} \frac{\tan(x)-x}{x^3} = \frac{0}{0} \overset{\text{l'Hop}}{=} \lim_{x\to 0} \frac{\sec^2(x)-1}{3x^2} = \frac{0}{0} \overset{\text{l'Hop again!}}{=} \lim \frac{2\sec(x)\cdot\sec(x)\tan(x)}{6x} = \frac{0}{0} \overset{\text{l'Hop again}}{=}$

$2\sec^2(x)\tan(x)$

$= \lim \frac{4\sec(x)\cdot\sec(x)\cdot\tan(x)\cdot\tan(x) + 2\sec^2\cdot\sec^2(x)}{6} = \frac{2}{6}$

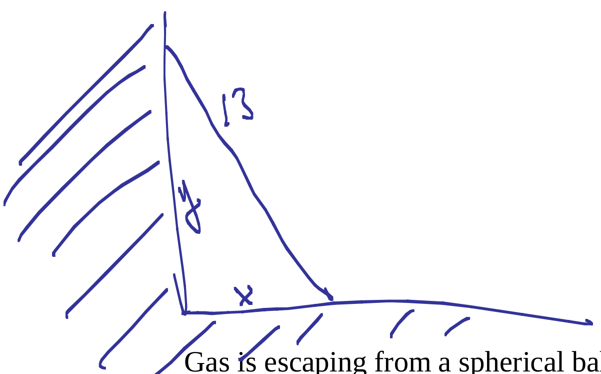
$= 1/3$

$\lim_{x\to 0} \frac{\cos(x^2)-1}{5x^4} = \frac{0}{0} =$ l'Hop several times $= -\frac{1}{10}$

$\lim_{x\to \pi} \frac{1+\cos(x)}{\sin(x)} = \frac{0}{0} = \lim \frac{-\sin(x)}{\cos(x)} = 0$

6. A 13 meter ladder is leaning against a wall If the top of the ladder slips down the wall at a rate of 2 m/sec, how fast will the foot be moving away from the wall when the top is 5 m above the ground.

$y' = -2$



$x^2 + y^2 = 13^2 = 169$

if $y = 5$ then $x^2 = 169 - 25 = 144$

$\Rightarrow x = 12$

$\cancel{2}xx' + \cancel{2}yy' = 0$

$x' = -yy'/x = -\frac{5\cdot(-2)}{12} = +5/6$ m/sec

Gas is escaping from a spherical balloon at a rate of 10 ft^3/hr. At what rate is the radius changing when the volume is 400 ft^3.

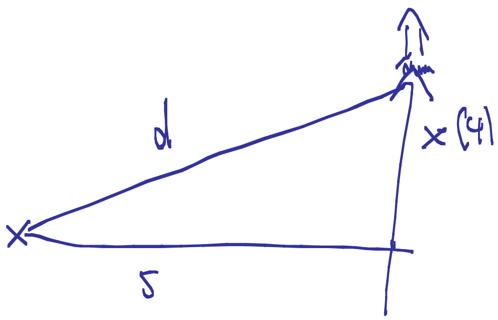
volume changes. $dV/dt = -10$ (neg. because volume gets smaller)

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \, dr/dt \Rightarrow dr/dt = \frac{dV/dt}{4\pi r^2} = \frac{-10}{4\pi r^2}$.

if $V = 400 = \frac{4}{3}\pi r^3 \Rightarrow \sqrt[3]{\frac{300}{\pi}} = r = 9.77$

$\Rightarrow dr/dt = \frac{-10}{4\pi\cdot(9.77)^2} = -0.008$ feet/sec

A radar station that is on the ground 5 miles from the launch pad tracks a rocket, rising vertically. How fast is this rocket rising when it is 4 miles high and its distance from the radar station is increasing at a rate of 2000 mph ?



$d'(t) = 2000$ mph

$d^2 = 25 + 16 = 41$

$x^2 + 25 = d^2$

$2xx' = 2dd'$

$\Rightarrow x' = dd'/x = \frac{\sqrt{41}\cdot 2000}{4} = 3201$ mph

7. Verify the linear approximation $\sqrt[3]{1-x} \approx 1 - \frac{1}{3}x$ near $c = 0$.

$f(x) = \sqrt[3]{1-x}$

$f'(x) = \frac{1}{3}(1-x)^{-2/3} \cdot (-1)$

$\Rightarrow f'(0) = -\frac{1}{3} \Rightarrow f(x) \approx f(0) - \frac{1}{3}x = 1 - \frac{1}{3}x$

Find the linear approximation of $\frac{1}{(1+2x)^4} \approx 1 - 8x$ near c = 0. Use Wolfram Alpha to graph both functions together to see if the approximation is indeed close.

$f(x) = (1+2x)^{-4}$ Find $f'(x)$ = as above!

Find the linearization of $f(x) = \frac{1}{\sqrt{2+x}}$ near $c = 0$. Do the same for $f(x) = x^4 + 3x^2$ near $c = -1$.

Similar

Use a linear approximation to estimate $(2.001)^5$. Do the ~~same for (8.006)~~$^{\frac{2}{3}}$.

$f(x) = x^5$ near $x = 2$

$f'(x) = 5x^4$ near $x = 2$; $f'(x) \approx 5x^4 \Rightarrow f'(2) = 5 \cdot 16 = 80$

$\Rightarrow \boxed{x^5 \approx 32 + 80(x-2)} \Rightarrow 2.001^5 \approx 32 + 80 \cdot 0.001 = 32.08$

8. The radius of a disk is given as 24 cm with a max. error of 0.2 cm. Use differentials to find the max. error of calculating the area of the disk as well as the relative error in percent.

$A = \pi r^2 \Rightarrow dA = 2\pi r\,dr = 2\pi \cdot 24 \cdot 0.2 \approx \underline{30.12}$

$\frac{dr}{r} = \frac{0.2}{24} = 0.008$ or 0.8% $\qquad A = \pi \cdot 24^2$

$\frac{dA}{A} = \frac{30.12}{24^2 \cdot \pi} = 0.016$ or $\underline{1.6\%}$

The edge of a cube was found to be 30 cm with a possible error of 0.1 cm. Use differentials to estimate the maximum possible error and the relative error in computing (a) the volume of the cube and (b) the surface area of the cube.

no more time

9. Graph the following functions, complete with domain, relative max/min, inflection points, asymptotes, etc. – the works

$f(x) = e^{-x^2}$

$f(x) = e^{-x^2}$

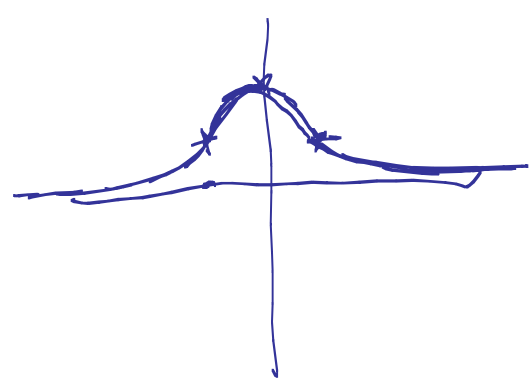
$\lim_{x\to\infty} e^{-x^2} = 0$ } horiz. a.s.

$\lim_{x\to-\infty} e^{-x^2} = 0$ no vertical a.s.

$f' = -2xe^{-x^2} \Rightarrow x = 0$ critical

$f'' = -2e^{-x^2} - 2xe^{-x^2}\cdot(-2x) = 2e^{-x^2}(2x^2-1) \Rightarrow x = \pm\sqrt{1/2}$

	$-\sqrt{1/2}$	0	$\sqrt{1/2}$	
f'	+	+	−	−
f''	+	−	−	+
f				



$f(x) = \frac{e^x}{x^2}$

no term

$f(x) = \frac{\tan^{-1}(x)}{1+x^2}$ no term

10. Find the inverse function for the following functions:

$f(x) = \frac{2x-1}{1-x}$ $\quad y = \frac{2x-1}{1-x} \Rightarrow y(1-x) = 2x-1$

$y - yx = 2x-1$

$y+1 = 2x+yx = x(2+y) \Rightarrow \frac{y+1}{2+y} = x \Rightarrow f^{-1}(x) = \frac{1+x}{2+x}$

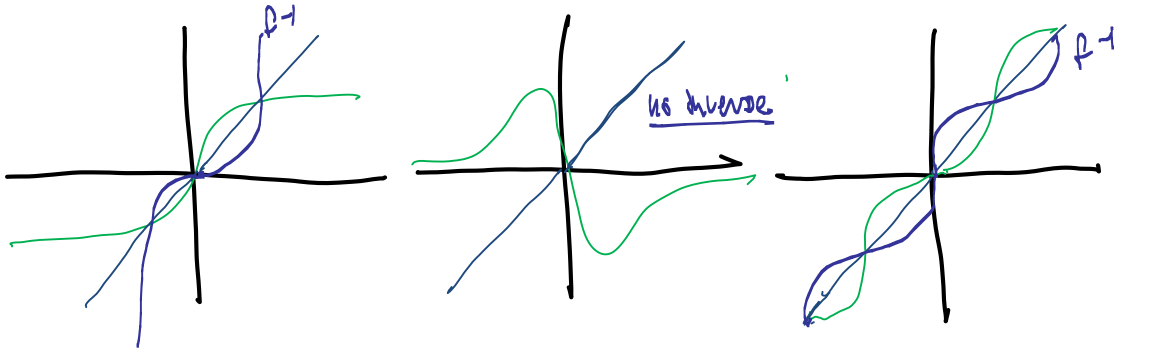
$f(x) = 2 - 3\sin(4x)$ $= y \Rightarrow y-2 = -3\sin(4x) \Rightarrow \frac{y-2}{-3} = \sin(4x) \Rightarrow \sin^{-1}\left(\frac{2-y}{3}\right) = 4x$

$\Rightarrow f^{-1}(x) = \frac{1}{4}\sin^{-1}\left(\frac{2-x}{3}\right)$

$f(x) = 3 \cdot 2^x - 8$ $= y \Rightarrow y+8 = 3\cdot 2^x \Rightarrow \frac{y+8}{3} = 2^x \Rightarrow x = \log_2\left(\frac{y+8}{3}\right)$

$\rightarrow f^{-1}(x) = \log_2\left(\frac{x+8}{3}\right)$

11. Which of the following functions have an *inverse* function? For those who do, sketch the inverse. Note that the blue line indicates the main diagonal, for reference.



$C(x) =$

EXTRA: Prove that $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$ for all *x*. *Hint: Find the derivative and interpret your answer*

$f'(x) = 0 \Rightarrow f(x) = \text{const.}$

$f(0) = 0 + \pi/2 = \pi/2$

$\Rightarrow \boxed{\sin^{-1}(x) + \cos^{-1}(x) = \pi/2}$

Prove that the derivative of the inverse sin function is $\frac{1}{\sqrt{1-x^2}}$ look it up!