Math 1401 - Practice Final Exam

This is a practice exam only. The actual exam may differ from this practice exam.
*In fact, there are **many more questions here than will be on the final exam**..*

Part 1: Integration Problems

1. Answer the following questions. Be concise:

a) State the *definition* as well as the *geometric interpretation* of:

- f(x) is continuous at the point x = a if

- (i) $f(a)$ exists (ii) $\lim_{x \to a} f(x)$ exists (iii) (i) = (ii)

Geometrically, the graph has no holes or gaps.

- the derivative of a function f(x), i.e. $\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Geometrically, it gives slope of tangent line.

- the definite integral of a function f(x) over an interval [a, b], i.e. $\int_a^b f(x)dx = \lim_{n \to \infty} \frac{1}{n} \left(f(x_1) + f(x_2) + \ldots + f(x_n) \right)$

Geometrically, it gives the (net) area under a curve.

b) What is the definition of the indefinite integral $\int f(x)\, dx$ = antiderivative F of, i.e. function F with $F'(x) = f(x)$

c) State the first fundamental theorem of calculus. What is that theorem good for, in your own words?

If f is continuous then $\int_a^b f(x)dx = F(b) - F(a)$, F antideriv. of f

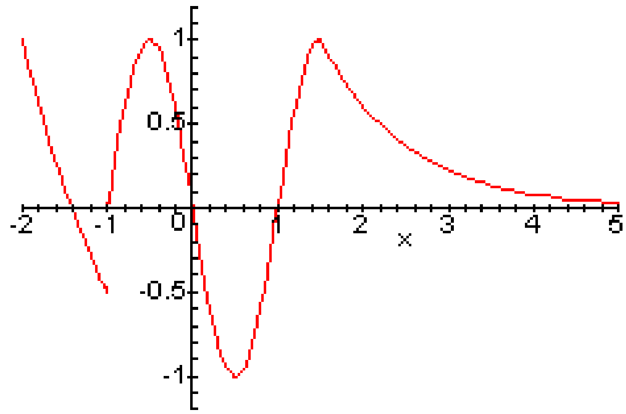
~~d) State the second fundamental theorem of calculus. What is that theorem good for, in your own words?~~

e) What is the difference between $\int_a^b f(x)dx$ and $\int f(x)dx$

number

function

2. Consider the function displayed below, and state whether the indicated quantities are positive, negative, or zero.



$\lim_{x\to -1^+} f(x) = 0$

$\lim_{x\to\infty} f(x) = 0$

f(0) = 0

f '(0) < 0 (neg)

f '(0.5) = 0

f ''(0.5) > 0 (pos)

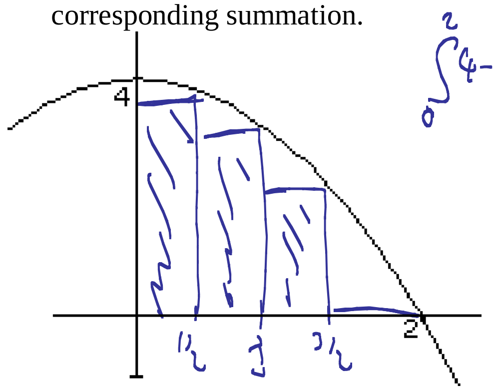
f''(3) > 0 (pos)

$\int_{-1}^{0} f(x)dx > 0$

$\int_{-1}^{1} f(x)dx = 0$

3. Consider the following definite integral: $\int_0^2 4 - x^2 dx$ (the integrand is depicted below).

a) Approximate the value of that definite integral by using 4 subdivisions and **right** rectangles in the corresponding summation.



$\int_0^2 4 - x^2\, dx \approx \frac{1}{2}\left(f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2)\right) =$

$= \frac{1}{2}\left((4 - (\tfrac{1}{2})^2) + (4 - 1^2) + (4 - (\tfrac{3}{2})^2) + (4 - 2^2)\right) =$

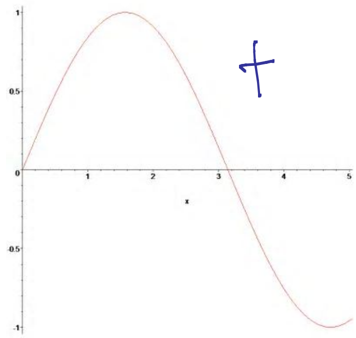
$= \frac{1}{2}\left(11 - \frac{1}{4} - \frac{9}{4}\right) = \frac{1}{2}\left(11 - \frac{10}{4}\right) = \frac{1}{2}\cdot\frac{34}{4} = \frac{17}{4} = 4.25$

b) Find the exact value of that definite integral by using the first fundamental theorem of calculus. Compare with the answer in (a).

$\int_0^2 4 - x^2 dx = 4x - \frac{1}{3}x^3\Big|_0^2 = \left(8 - \frac{8}{3}\right) - 0 = \frac{16}{3} = 5.33$

The exact value is larger than the estimation

4. Below are the graphs of four functions. For each graph, decide whether $\int_0^5 f(x)dx$ is positive, negative, or zero.



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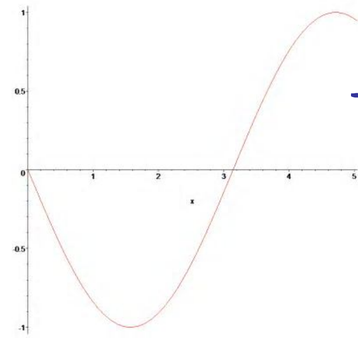

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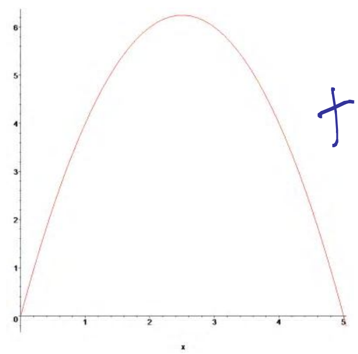

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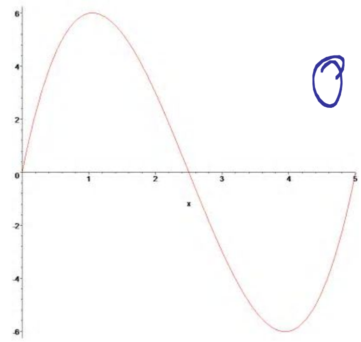

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5. Find the integrals (definite or indefinite) in each of the following problems, using any method you like.

$\int 3x^2 - \frac{5}{x^3} + \sqrt[3]{x} - 2\cos(x)dx$ $= x^3 + 5\frac{1}{2}x^{-2} + \frac{3}{4}x^{4/3} - 2\sin(x) + C$

$\int x^2 + \frac{1}{\sqrt{x}} + \pi^2 dx$ $= \frac{1}{3}x^3 + 2x^{1/2} + \pi^3 x + C$

$\int (x+\frac{1}{x})^2 dx$ $= \int x^2 + 2x\frac{1}{x} + \frac{1}{x^2}dx = \int x^2 + 2 + x^{-2}dx = \frac{1}{3}x^3 + 2x - x^{-1} + C$

$\int \frac{5}{1+x^2}dx$ $= 5\cdot\tan^{-1}(x) + C$

$\int x^2 + \frac{1}{x^3} + \frac{1}{x} - \cos(x) + e^x + \pi^2 dx$ $= \frac{1}{3}x^3 - \frac{1}{2}x^{-2} + \ln(x) - \sin(x) + e^x + \pi^3 x + C$

$$\int 5e^x - \frac{7}{3\sqrt{1-x^2}}dx = 5e^x - \frac{7}{3}\sin^{-1}(x) + c$$

$$\int \frac{2-t^3}{\sqrt{t}}dt = \int \frac{2}{\sqrt{t}} - \frac{t^3}{\sqrt{t}}\,dt = \int 2\cdot t^{-1/2} - t^{5/2}\,dt = 2\cdot\frac{2}{1}t^{1/2} - \frac{2}{7}t^{7/2} + c$$

subst

$$\int_0^1 x^2 e^{-x^3}dx = -\frac{1}{3}\int_{x=0}^{x=1} e^u\,du = -\frac{1}{3}e^u\Big|_{x=0}^{x=1} = -\frac{1}{3}e^{-x^3}\Big|_0^1 = -\frac{1}{3}e^{-1} + \frac{1}{3} = \frac{1}{3}\left(1-\frac{1}{e}\right)$$

$u = -x^3 \Rightarrow du = -3x^2dx$ so that $-\frac{1}{3}du = x^2dx$

subst

$$\int_0^1 \frac{2x-4}{x^2-4x+5}dx = \int_{x=0}^{x=1}\frac{1}{u}du = \ln(u)\Big|_{x=0}^{x=1} = \ln(x^2-4x+5)\Big|_0^1 = \ln(1-4+5) - \ln(5) = \ln(2) - \ln(5)$$

$u = x^2-4x+5 \Rightarrow du = 2x-4\,dx$

subst

$$\int \frac{e^x}{2+e^x}dx = \int \frac{1}{u}du = \ln|u| + c = \ln(2+e^x) + c$$

$u = 2+e^x \Rightarrow du = e^x dx$

subst

$$\int \frac{\ln(x)}{x}dx = \int u\,du = \frac{1}{2}u^2 + c = \frac{1}{2}(\ln(x))^2 + c$$

$u = \ln(x) \Rightarrow du = \frac{1}{x}dx$

$$\int \frac{1}{x\ln(x)}dx = \int \frac{1}{u}du = \ln|u| + c = \ln(\ln(x)) + c$$

$u = \ln(x) \Rightarrow du = \frac{1}{x}dx$

$$\int \frac{\ln(e^{x^2})}{e^{2\ln(x)}}dx = \int \frac{x^2}{x^2}dx = \int 1\,dx = x + c$$

$$\int_1^3 \frac{1}{t^2} + t\,dt = -t^{-1} + \frac{1}{2}t^2\Big|_1^3 = \left(-3^{-1} + \frac{1}{2}3^2\right) - \left(-1 + \frac{1}{2}\right) = \frac{14}{3}$$

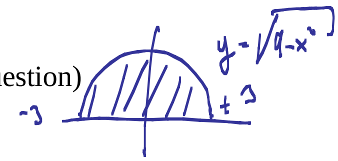
$$\int_1^1 \sin(x)e^{x^2}dx = 0$$ (no area under curve from 1 to 1)

$$\int_1^1 \cos(x)e^{-x^3}dx = 0$$

$$\int_0^{\pi/2} \cos(x) - 3\sin(x)dx = \sin(x) + 3\cos(x)\Big|_0^{\pi/2} = \left[\sin\left(\tfrac{\pi}{2}\right) + 3\cos\left(\tfrac{\pi}{2}\right)\right] - \left[\sin(0) + 3\cos(0)\right] = 1 + 0 - (0+3) = \underline{\underline{-2}}$$

$$\int_1^2 x(x-1)dx = \int_1^2 x^2 - x\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2\Big|_1^2 = \left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \underline{\underline{\frac{7}{3} - \frac{3}{2}}}$$

$\int_{-3}^{3} \sqrt{9-x^2}\,dx$ (trick question)



$= \frac{1}{2}$ area of circle radius $3 = \frac{1}{2}\cdot\pi\, 3^2 = \underline{\underline{\frac{9}{2}\pi}}$

$$\int_1^4 \frac{x-1}{\sqrt{x}}dx = \int_1^4 \frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}dx = \int_1^4 x^{1/2} - x^{-1/2}dx = \frac{2}{3}x^{3/2} - 2x^{1/2}\Big|_1^4 = \left(\frac{2}{3}4^{3/2} - 2\cdot 4^{1/2}\right) - \left(\frac{2}{3} - 2\right)$$
$$= \frac{2}{3}\cdot 8 - 4 - \frac{2}{3} + 2 = \underline{\underline{\frac{14}{3} - 2}}$$

$$\int_1^2 \frac{2-t^3}{t^2}dt = \int_1^2 \frac{2}{t^2} - \frac{t^3}{t^2}dt = \int_1^2 2t^{-2} - t\,dt = 2(-1)t^{-1} - \frac{1}{2}t^2\Big|_1^2$$
$$= \left(-2\cdot 2^{-1} - \frac{1}{2}2^2\right) - \left(-2 - \frac{1}{2}\right) = -1 \cancel{-2} \cancel{+2} + \frac{1}{2} = -\frac{1}{2}$$

$\int_{\pi}^{\pi} \cos(4x^2)dx$ (trick questi $= \underline{\underline{0}}$

6. ~~Define a function~~ $\cancel{Erf(x) = \int_0^x e^{-t^2}dt}$ ~~. Then find S(0), find S'(0), and find S"(0)~~

7. ~~Define a function~~ $\cancel{S(x) = \int_0^x \cos^2(t^2)dt}$ ~~. Then find S(0), S'(x), and S"(x).~~

Part II. Review of previous concepts

Evaluate the following limits (if necessary and appropriate, use l'Hopital's Rule):

$$\lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0$$

$$\lim_{x\to 0} \frac{x}{x^2-1} = \frac{0}{-1} = 0$$

$$\lim_{x\to 2} \frac{x-2}{x^2+x-6} = \frac{0}{0} = \lim_{x\to 2} \frac{1}{2x+1} = \underline{\underline{1/5}}$$

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \frac{0}{0} = \lim_{x\to 2}\frac{2x}{1} = \underline{4}$$

$$\lim_{x\to\infty}\frac{x^2-3x^3+9}{2x^3-6} = 0$$

$$\lim_{x\to\infty}\frac{x^2-3x^3+9}{2x^4-6} = 0$$

$$\lim_{x\to-\infty}\frac{x^2-3x^3+9}{2x^2-6} = \pm\infty$$

$$\lim_{x\to 0^+} x\ln(x) \left[= 0\cdot(-\infty)\right] = \lim_{x\to 0^+}\frac{\ln(x)}{1/x} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = \underline{\underline{0}}$$ (very tricky)

$$\lim_{x\to 2}\frac{x}{x-2} = \underline{\text{undef}}$$

$$\lim_{x\to 2}\frac{x-2}{x^2+x-6} = \frac{0}{0} = \lim_{x\to 2}\frac{1}{2x+1} = 1/5$$

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \underline{\underline{4}}$$

$$\lim_{x\to 0}\frac{\sin(3x)}{5x} = \frac{0}{0} = \lim_{x\to 0}\frac{3\cos(3x)}{5} = \underline{\underline{3/5}}$$

$$\lim_{x\to 0}\frac{\sin(3x)}{\sin(7x)}$$ or $= \left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{3\cos(3x)}{7\cos(7x)} = \underline{\underline{3/7}}$

$$\lim_{x\to 0}\frac{\cos(3x)-1}{x^2} = \left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-3\sin(3x)}{2x} = \left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-9\cos(3x)}{2} = \underline{-9/2}$$

$$\lim_{x\to\infty}\frac{x^2-3x^3+9}{2x^3-6} = \underline{\underline{-\frac{3}{2}}}$$

$$\lim_{x\to\infty}\frac{x^2-3x^3+9}{2x^4-6} = 0$$

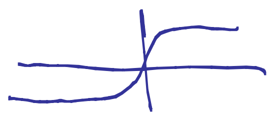
$$\lim_{x\to-\infty}\frac{x^2-3x^3+9}{2x^2-6} = \text{undef}$$

$$\lim_{x\to\infty}\frac{x^2}{e^x} \text{ or } \left(=\frac{\infty}{\infty}\right) = \lim_{x\to\infty}\frac{2x}{e^x} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty}\frac{2}{e^x} = 0$$

$$\lim_{x\to0}\frac{\sin(x)-x}{x^3}\left(=\frac{0}{0}=\right)\lim\frac{\cos(x)-1}{3x^2}\left(=\frac{0}{0}\right)\lim_{x\to0}\frac{-\sin(x)}{6x} = \left(\frac{0}{0}\right) = \lim_{x\to0}\frac{-\cos(x)}{6} = -1/6$$

$$\lim_{x\to0^+} x\ln(x) = 0 \text{ (see above)}$$

$$\lim_{x\to\infty}\arctan(x) = \pi/2$$



$$\lim_{x\to\infty} e^{-x^2} = \lim\frac{1}{e^{x^2}} = 0$$

$$\lim_{x\to1}\frac{\ln(x)}{x-1} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to1}\frac{1/x}{1} = 1$$

$$\lim_{x\to0}\frac{\tan(x)-x}{x^3} = \left(\frac{0}{0}\right) = \lim_{x\to0}\frac{\sec^2(x)-1}{3x^2} = \left(\frac{0}{0}\right) = \lim\frac{2\sec(x)\cdot\sec(x)\cdot\tan(x)}{6x} = \lim\frac{2\sec^2(x)\tan(x)}{6x}$$

$$= \lim_{x\to0}\frac{4\sec(x)\cdot\sec(x)\tan(x)\tan(x)+2\sec^2(x)\cdot\sec^2(x)}{6} = \frac{2}{6} = \frac{1}{3}$$

$$\lim_{x\to0}\frac{\cos(x^2)-1}{5x^4} = \left(\frac{0}{0}\right) = \lim_{x\to0}\frac{-2x\sin(x^2)}{20x^3} = \dots = -\frac{1}{10}$$

$$\lim_{x\to\pi}\frac{\sin(x)}{1-\cos(x)} = \frac{0}{2} = 0$$

$$\lim_{x\to0^+} x\ln(x) = 0 \text{ (see above)}$$

$$\lim_{x\to\frac{\pi}{2}^+}\sec(x)-\tan(x) = \lim_{x\to\pi/2}\frac{1}{\cos(x)}-\frac{\sin(x)}{\cos(x)} = \lim_{x\to\pi/2}\frac{1-\sin(x)}{\cos(x)}\left(=\frac{0}{0}\right) = \lim_{x\to\pi/2}\frac{-\cos(x)}{-\sin(x)} = -1/-1 = 1$$

Find the number k, if any, so that the following function is continuous:

$$f(x)=\begin{cases}\dfrac{x^2+3x-10}{x-2} & \text{if } x\neq 2\\ k & \text{if } x=2\end{cases}$$

(i) $f(2)=k$ (ii) $\lim\limits_{x\to 2} f(x)=\lim\limits_{x\to 2}\dfrac{x^2+3x-10}{x-2}=\lim\limits_{x\to 2}\dfrac{2x+3}{1}=7 \Rightarrow \underline{k=7}$

Use the ***definition*** of derivative to find the derivative of $f(x)=x^2+2x-2$.

$$f'(x)=\lim_{h\to 0}\frac{[(x+h)^2+2(x+h)-2]-[x^2+2x-2]}{h}=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2+\cancel{2x}+2h-\cancel{2}-\cancel{x^2}-\cancel{2x}+\cancel{2}}{h}$$

$$=\lim_{h\to 0}\frac{h(2x+h+2)}{h}=\underline{2x+2}$$

Do the same for $f(x)=\frac{1}{1-x}$.

You do it

Find the derivatives of the following functions, using any method you like.

$f(x)=x^2+\ln(x)+e^x+\pi^2$ $\quad f'=2x+\frac{1}{x}+e^x$

$f(x)=(5x^2+2)^3\cos^2(x)$ $\quad f'=3(5x^2+2)^2\cos^2(x)+(5x^2+2)^3\cdot 2\cos(x)(-\sin(x))$

$f(x)=\ln(x^3)$ $\quad f'(x)=\frac{1}{x^3}\cdot 3x^2=\frac{3}{x}$

$f(x)=e^{-5x^2}$ $\quad f'(x)=-10xe^{-5x^2}$

$f(x)=\dfrac{\ln(2x+1)}{x^2+3}$ $\quad f'(x)=\dfrac{\frac{2}{2x+1}(x^2+3)-\ln(2x+1)(2x)}{(x^2+3)^2}$

$f(x)=\dfrac{\ln(\sin(x))}{e^{-x^2}}$ $\quad f'(x)=\dfrac{\frac{1}{\sin(x)}\cdot\cos(x)\cdot e^{-x^2}+\ln(\sin(x))\,2xe^{-x^2}}{(e^{-x^2})^2}$

$f(x)=\ln\left(\dfrac{(x+1)^2(1-x)^3}{\cos^3(x)e^x}\right)=2\ln(x+1)+3\ln(1-x)-3\ln(\cos(x))-\cancel{\ln(e^x)}\,x$

$\Rightarrow f'(x)=\dfrac{2}{x+1}+\dfrac{3}{1-x}(-1)-\dfrac{3}{\cos(x)}\cdot(-\sin(x))-1$

$f(x) = \tan(x^3)$ $f' = \sec^2(x^3) \cdot 3x^2$

$f(x) = \cos(\cos^{-1}(x)) - e^{\ln(x)}$ $= x - x = 0 \Rightarrow \underline{f'(x) = 0}$

$f(x) = \sin^{-1}(2x) + 3\sec(4x)$ $f' = \frac{2}{\sqrt{1-4x^2}} + 3 \cdot 4 \cdot \sec(4x)\tan(4x)$

$f(x) = \arctan(x^2)(1 + x^2)^3$ $f' = \frac{2x}{1+x^4}(1+x^2)^3 + \arctan(x^2)\,3(1+x^2)^2 \cdot 2x$

$f(x) = \frac{(x-1)^2}{(x+1)^3}(x+2)^4$

use log differentiation

$g(x) = \frac{\sqrt{x^2-1}}{x^5(x-4)^4}$

Sketch functions like $f(x) = \frac{x^2}{x^2-1}$ or $f(x) = \frac{x}{x^2-1}$, or $f(x) = e^{-x^2}$. Or, sketch the function $f(x) = \frac{\ln(x)}{x}$ or $f(x) = \frac{\ln(x)}{x^2}$ (asymptotes x = 0 and y = 0).

follow the usual recipe

Find all *relative* extrema of the function $f(x) = e^{-x^2}$ and classify them, if any.

$f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$

$\Rightarrow x = 0$ is critical

	-1	0	1
f'	+		−
f	↗		↘

$x = 0$ gives a local max

Part III: Story Problems

- Bacteria follows exponential growth. Initially there are 100 cells but after one hour the population has increased to 420. Find the number of bacteria after 3 hours. Also find when the population reaches 10,000 cells.

Model: $P(t) = P_0 e^{kt}$

$P(0) = P_0 = 100$

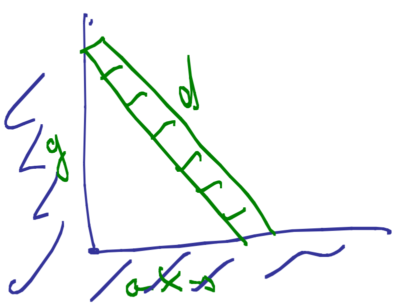
$P(1) = 100e^k = 420$

$\Rightarrow e^k = \frac{420}{100} = 4.2 \Rightarrow k = \ln(4.2) = 1.435$

$\Rightarrow$ after 3 hours: $P(3) = 100e^{1.435 \cdot 3} = 7408$

also: $10000 = 100e^{1.435 \cdot t} \Rightarrow \frac{\ln(100)}{1.435} = t = 3.21$ hours

- A 13 meter ladder is leaning against a wall If the top of the ladder slips down the wall at a rate of 2 m/sec, how fast will the foot be moving away from the wall when the top is 5 m above the ground.



$d^2 = x^2 + y^2 \Rightarrow 169 = x^2 + y^2$

$\Rightarrow 0 = \cancel{2}xx' + \cancel{2}yy'$

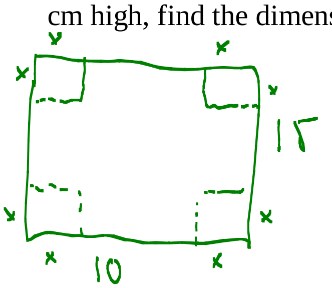
$\Rightarrow x' = \frac{-y \cdot y'}{x} = \frac{-12(-2)}{5} = \frac{24}{5}$

if $x = 5 \Rightarrow 169 = 25 + y^2$

$\Rightarrow y = 12$

Know: $y' = -2$

- A square is cut out from each corner of a rectangular piece of cardboard. The remaining sides of the cardboard are folded up to form a box without lid. If the original cardboard piece was 10 cm wide and 15 cm high, find the dimensions of the largest box possible.



$V = x \cdot (10 - 2x)(15 - 2x) = 150x - 30x^2 - 20x^2 + 4x^3$

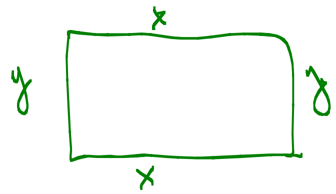
$A = 150x - 50x^2 + 4x^3, \quad x \in [0, 5]$

$A' = 150 - 100x + 12x^2 = 0 \Rightarrow x = \frac{5}{6}(5 \pm \sqrt{7})$

One is too big so $x = \frac{5}{6}(5 - \sqrt{7}) \approx 1.96$

- A Rectangular plot of land is fenced in with two kinds of fencing on opposite ends. Type A costs $3 per foot, Type B costs $2 per foot. Maximize the area if you have $6000.00 available. x = one side of rectangle with type A fencing, y = other side of rectangle with type B fencing. Setting up the basic equations for area and cost of fencing.

$A = xy$



Cost: $6000 = 3(2x) + 2(2y) = 6x + 4y$

$\Rightarrow 150 = \frac{6}{4}x + y \Rightarrow y = 150 - \frac{6}{4}x$

$\Rightarrow A = x\left(150 - \frac{6}{4}x\right) = 150x - \frac{6}{4}x^2$

$\Rightarrow A' = 150 - \frac{6}{2}x = 0 \Rightarrow \underline{\underline{x}} = 150 \cdot \frac{2}{6} = \underline{\underline{50}}$

- You are standing on a bridge that is 30 m above a river, and you are dropping a ball into the river. The velocity of the ball is given by the equation v(t) = 10 t. We know that the velocity is the derivative of the distance function, hence the distance function is the antiderivative of the velocity function. Use that information to determine the distance function. Then use that distance function to determine when the ball hits the water.
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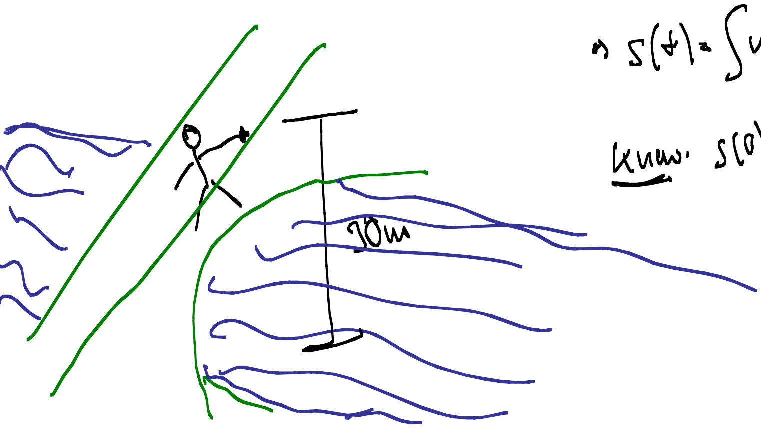

$v(t) = -10t$

$\Rightarrow s(t) = \int v(t)\,dt = -5t^2 + C$

Know: $s(0) = 30 = \underline{\underline{C}}$

so: $\underline{\underline{s(t) = -5t^2 + 30}}$

Hits ground when $s(t) = -5t^2 + 30 = 0$

$t^2 = \frac{30}{5} = 6$

$\underline{\underline{t = \sqrt{6}}}$ time to hit the water.