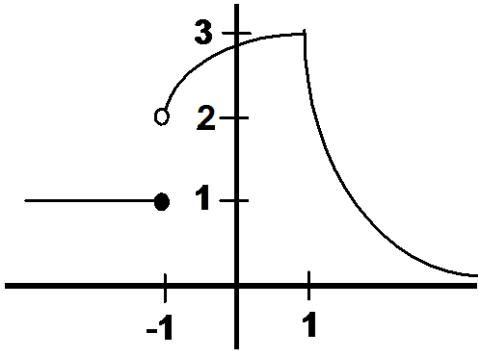


Calculus 1501: Practice Exam 1

1. State the following definitions or theorems:
 - a) Definition of a function $f(x)$ having a limit L
 - b) Definition of a function $f(x)$ being continuous at $x = c$
 - c) Definition of the derivative $f'(x)$ of a function $f(x)$
 - d) The "Squeezing Theorem"
 - e) The "Intermediate Value Theorem"
 - f) Theorem on the connection of differentiability and continuity
 - g) Derivatives of $\sin(x)$ and $\cos(x)$ (with proofs)

2. The picture on the left shows the graph of a certain function. Based on that graph, answer the questions:



- a) $\lim_{x \rightarrow -1^-} f(x)$ **1**
- b) $\lim_{x \rightarrow -1^+} f(x)$ **2**
- c) $\lim_{x \rightarrow -1} f(x)$ **3**
- d) $\lim_{x \rightarrow 0} f(x)$ **~ 2.7**
- e) Is the function continuous at $x = -1$? **No**
- f) Is the function continuous at $x = 1$? **YES**
- g) Is the function differentiable at $x = -1$? **No**
- h) Is the function differentiable at $x = 1$? **No**
- i) Is $f'(0)$ positive, negative, or zero? **pos**
- k) What is $f'(-2)$? **0**

3. Find each of the following limits (show your work):

a) $\lim_{x \rightarrow 3} 4\pi$ **4π**

b) $\lim_{x \rightarrow 3} \frac{x^2 - 2x}{x + 3} = \frac{9 - 6}{6} = \frac{1}{2}$

c) $\lim_{x \rightarrow 3} \frac{3 - x}{x^2 + 2x - 15} = \lim_{x \rightarrow 3} \frac{-(x-3)}{(x-3)(x+5)} = \frac{-1}{8}$

d) $\lim_{x \rightarrow 1^+} \frac{x}{x-1} \sim \frac{1}{1-0} = \infty$

e) $\lim_{x \rightarrow 1^-} \frac{x}{x-1} \sim \frac{1}{-0} = -\infty$

f) $\lim_{x \rightarrow 1} \frac{x}{x-1}$ **d.u.e.**

g) $\lim_{x \rightarrow 0} \frac{\sin^2(x)}{3x^2} = \frac{1}{3}$
 $\lim_{x \rightarrow 0} \frac{1}{3} \cdot \frac{\sin(x)}{x} \cdot \frac{\sin(x)}{x}$

h) $\lim_{x \rightarrow 0} \frac{\sin^2(x)}{\cos^2(x)} = 0$

i) $\lim_{x \rightarrow 0} \frac{\sin(6x)}{7x} = \lim_{x \rightarrow 0} \frac{1}{7} \cdot 6 \cdot \frac{\sin(6x)}{6x} = \frac{6}{7}$

j) $\lim_{t \rightarrow 0} \frac{t^2}{1 - \cos(t)}$

k) $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$
squeeze

l) $\lim_{x \rightarrow -\infty} \frac{3x^2 - 1}{2 - 3x - 4x^2} = -\frac{3}{4}$

m) $\lim_{x \rightarrow -\infty} \frac{3x^2 - 1}{2 - 3x} = +\infty$
 $\lim_{x \rightarrow -\infty} \frac{3x^2}{-3x} = +\infty$

n) $\lim_{x \rightarrow \infty} \sqrt{x^2 - 1} - x = \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1} + x} \cdot \frac{(x^2 - 1) - x^2}{\sqrt{x^2 - 1} + x} = \frac{-1}{\sqrt{x^2 - 1} + x} \rightarrow 0$

4. Consider the following function: $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ x - 2, & \text{if } x < 0 \end{cases}$
 $\lim_{x \rightarrow 0^-} f(x) = -2$
 $\lim_{x \rightarrow 0^+} f(x) = 0$
 $\lim_{x \rightarrow 0} f(x)$ **d.u.e.**

a) Find $\lim_{x \rightarrow 0^-} f(x) = -2$

b) Find $\lim_{x \rightarrow 0^+} f(x) = 0$

c) Find $\lim_{x \rightarrow 2} f(x)$ (note that x approaches two, not zero)

d) Is the function continuous at $x = 0$

No!

f) Is $f(x) = \begin{cases} \frac{x^2 - 1}{x + 1}, & \text{if } x \neq -1 \\ 17, & \text{if } x = -1 \end{cases}$ continuous at -1 ? If not, is the discontinuity removable?

$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} = \lim_{x \rightarrow -1} \frac{(x-1)(x+1)}{x+1} = -2 \neq f(-1) = 17$

so not continuous at $x = -1$. Removable!

g) Is there a value of k that makes the function g continuous at $x = 0$? If so, what is that value?

$g(x) = \begin{cases} x - 2, & \text{if } x \leq 0 \\ k(3 - 2x) & \text{if } x > 0 \end{cases}$

$\lim_{x \rightarrow 0^-} f(x) = -2$, $\lim_{x \rightarrow 0^+} f(x) = 3k$ want them to be equal, so

set $k = -\frac{2}{3}$

5. Please find out where the following functions are continuous:

a) $f(x) = \cos(x^2 - 2)$

everywhere

b) $f(x) = \frac{x}{1 - \sin^2(x)}$

all $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$

c) $f(x) = \begin{cases} \frac{\sin^2(x)}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ for all x

$\lim_{x \rightarrow 0} \frac{\sin^2(x)}{x} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \cdot \sin(x) = 1 \cdot 0 = 0 = f(0)$

d) $f(x) = \begin{cases} \frac{\sin(x)}{2x}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$

$\lim_{x \rightarrow 0} \frac{\sin(x)}{2x} = \frac{1}{2} \neq f(0)$ so not cont. at $x = 0$

6. Find the value of k , if any, that would make the following function continuous at $x = 4$.

$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases}$

$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$ - so make $k = 4$

7. Prove that the function $x^3 - 4x + 1 = 0$ has at least one solution in the interval $[1, 2]$. Also, prove that the function $x = \cos(x)$ has at least one solution in the interval $[0, \pi/2]$

$f(x) = x^3 - 4x + 1$. $f(1) = -2 < 0$, $f(2) = 1 > 0$. Since f is also cont. we can use Intermediate Value Theorem to conclude that there is at least one c in $(1, 2)$ with $f(c) = 0$

$x = \cos(x) \Leftrightarrow x - \cos(x) = 0$. Let $g(x) = x - \cos(x)$. Then g is cont. and $g(0) = -\cos(0) = -1 < 0$ and $g(\pi/2) = \pi/2 - 0 = \pi/2 > 0$. Thus, by Int. Value Theorem $g(c) = 0$ for some $c \in (0, \pi/2)$.

8. Use the *definition* of derivative to find the derivative of the function $f(x) = 3x^2 + 2$. Note that we of course know by our various shortcut rules that the derivative is $f'(x) = 6x$. Do the same for the function

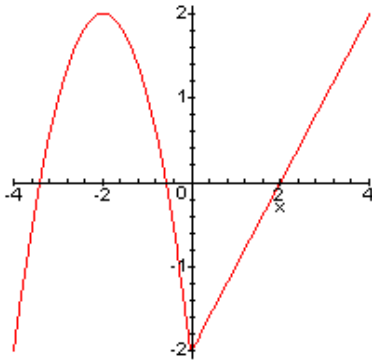
$f(x) = \frac{1}{1-x}$ and for $f(x) = \sqrt{x}$ (use definition!)

$$\begin{aligned} \underline{f(x) = 3x^2 + 2} : f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{3(x+h)^2 + 2 - (3x^2 + 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3(x^2 + 2xh + h^2) + 2 - 3x^2 - 2}{h} = \lim_{h \rightarrow 0} \frac{6xh + 3h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(6x + 3h)}{h} = \underline{\underline{6x}} \end{aligned}$$

$$\begin{aligned} f(x) = \frac{1}{1-x} : f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{1-(x+h)} - \frac{1}{1-x}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \frac{(1-x) - (1-(x+h))}{(1-(x+h))(1-x)} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{\cancel{1-x} - \cancel{1-x} + h}{(1-(x+h))(1-x)} = \lim_{h \rightarrow 0} \frac{1}{(1-(x+h))(1-x)} \\ &= \underline{\underline{\frac{1}{(1-x)^2}}} \end{aligned}$$

$f(x) = \sqrt{x}$: did in class (hint: use conjugate)

9. Consider graph of $f(x)$ you see below, and find the sign of the indicated quantity, if it exists. If it does not exist, please say so.



$$f(0) = -2 < 0$$

$$f'(0) \text{ undef.}$$

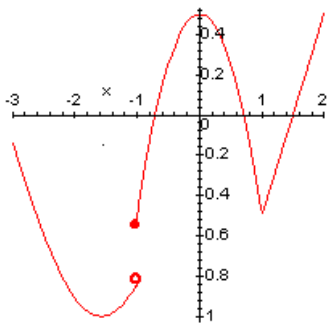
$$f(-2) = 2$$

$$f'(-2) = 0$$

$$f(2) = 0$$

$$f'(2) > 0$$

10. Consider the function whose graph you see below, and find a number $x = c$ such that



- a) f is not continuous at $x = a$ at $x = -1$
 b) f is continuous but not differentiable at $x = b$ at $x = 1$
 c) f' is positive at $x = c$ at $x = 0$ or $x = -0.5$
 d) f' is negative at $x = d$ at $x = -3$ or $x = 0.5$
 e) f' is zero at $x = e$ at $x = 0$
 f) f' does not exist at $x = f$ at $x = -1$ and $x = 1$

10. Please find the derivative for each of the following functions (do not simplify unless you think it is helpful).

$$f(x) = \pi^2 + x^2 + \sin(x) + \sqrt{x}$$

$$f'(x) = 0 + 2x + \cos(x) + \frac{1}{2}x^{-1/2}$$

$$f(x) = x^2(x^4 - 2x) = x^6 - 2x^3 \Rightarrow f'(x) = 6x^5 - 6x^2$$

$$f(x) = x^2\left(x^3 - \frac{1}{x}\right) = x^5 - x \Rightarrow f'(x) = 5x^4 - 1$$

$$f(x) = 3x^5 - 2x^3 + 5x - \sqrt{2}$$

$$\Rightarrow f'(x) = 15x^4 - 6x^2 + 5 - 0$$

$$f(x) = \frac{x^4 - 2x + 3}{x^2} = x^2 - \frac{2}{x} + \frac{3}{x^2} \Rightarrow f'(x) = 2x + 2x^{-2} - 6x^{-3}$$

$$f(x) = x^3 \sin(x) \quad f'(x) = 3x^2 \sin(x) + x^3 \cos(x)$$

$$f(x) = \sin(x) \cos(x) \quad f'(x) = \cos(x) \cos(x) + \sin(x) (-\sin(x)) \\ = \cos^2(x) - \sin^2(x)$$

$$f(x) = \sin^2(x) = \sin(x) \cdot \sin(x) \Rightarrow f'(x) = \cos(x) \sin(x) + \sin(x) \cos(x) = 2 \cos(x) \sin(x)$$

$$f(x) = \frac{\sin(x)}{x^4 - 3} \quad f'(x) = \frac{\cos(x)(x^4 - 3) - \sin(x) \cdot 4x^3}{(x^4 - 3)^2}$$

$$f(x) = \frac{\sec(x)}{x^4} \quad f'(x) = \frac{\sec(x) \tan(x) x^4 - \sec(x) 4x^3}{x^8} \quad \text{because } (\sec)' = \left(\frac{1}{\cos}\right)' = \frac{0 \cdot \cos - 1 \cdot (-\sin)}{\cos^2} \\ = \sin / \cos^2 = \sec \cdot \tan$$

$$f(x) = \tan(x) \sqrt{x} \quad f'(x) = \sec^2(x) \sqrt{x} + \tan(x) \frac{1}{2} x^{-1/2} \quad \text{because } (\tan)' = \left(\frac{\sin}{\cos}\right)' = \frac{\cos^2 + \sin^2}{\cos^2} = \sec^2$$

$$f(x) = \pi^2 \sin\left(\frac{\pi}{6}\right) \quad f'(x) = 0!$$

$$f(x) = \frac{x^4 - 2x + 3}{x^2 - 4x} \quad f'(x) = \frac{(4x^3 - 2)(x^2 - 4x) - (x^4 - 2x + 3)(2x - 4)}{(x^2 - 4x)^2}$$

$$f(x) = \frac{x^2}{x^2 - 1} \quad f'(x) = \frac{2x(x^2 - 1) - x^2(2x)}{(x^2 - 1)^2}$$

$$f(x) = \frac{x \sin(x)}{x - 3} \quad f'(x) = \frac{(1 \cdot \sin(x) + x \cos(x))(x - 3) - x \sin(x) \cdot 1}{(x - 3)^2}$$

Pluchelows!

$$f(x) = \frac{x^2 \cos(x)}{(1-2x)\sin(x)}$$

$$f'(x) = \frac{(2x \cos(x) + x^2(-\sin(x)))(1-2x)\sin(x) - x^2 \cos(x)[(-2)\sin(x) - (1-2x)\cos(x)]}{(1-2x)^2 \sin^2(x)}$$

$$f(x) = \tan(x), \text{ find } f''(x)$$

$$f'(x) = \sec^2(x) = \sec(x)\sec(x) \rightarrow$$

$$f''(x) = [\sec(x)\tan(x)]\sec(x) + \sec(x)[\sec(x)\tan(x)] = 2\sec^2(x)\tan(x)$$

$$f(x) = x \cos(x), \text{ find } f'''(x)$$

$$f' = 1 \cdot \cos(x) - x \sin(x), \quad f'' = -\sin(x) - (1 \sin(x) + x \cos(x)) = -2\sin(x) - x \cos(x)$$

$$f''' = -2\cos(x) - (1 \cdot \cos(x) - x \sin(x)) = -3\cos(x) + x \sin(x)$$

$$f(x) = 3x^5 - 2x^3 + 5x - 1, \text{ find } f^{(7)}(x)$$

$$f^{(7)}(x) = 0$$

11. Find the equation of the tangent line to the function at the given point:

a) $f(x) = x^2 - x + 1$, at $x = 0$

$$f'(x) = 2x - 1 \rightarrow f'(0) = -1 \Rightarrow \text{slope} = -1$$

$$f(0) = 1 \Rightarrow$$

$$\underline{y - 1 = -1(x - 0)}$$

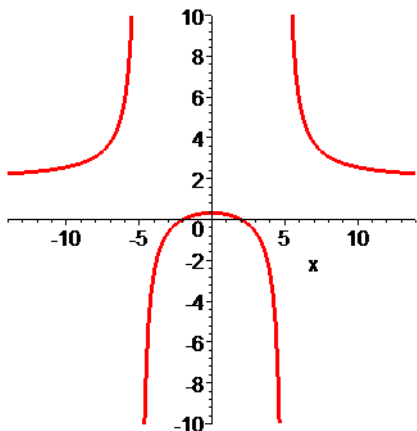
b) $f(x) = x^3 - 2x$, at $x = 1$

$$f'(x) = 3x^2 - 2 \rightarrow f'(1) = 3 - 2 = 1 \text{ so slope}$$

$$f(1) = -1 \Rightarrow$$

$$\underline{y + 1 = 1(x - 1)}$$

12. For the function displayed below, find the following limits:



a) $\lim_{x \rightarrow \infty} f(x) = 2$

b) $\lim_{x \rightarrow -\infty} f(x) = 2$

c) $\lim_{x \rightarrow 5^+} f(x) = +\infty$

d) $\lim_{x \rightarrow -5^+} f(x) = -\infty$

12. Suppose the function $f(x) = \frac{x^4 - 2x + 3}{x^2}$ indicates the position of a particle. $f(t) = t^2 - \frac{2}{t} + \frac{3}{t^2}$

a) Find the velocity after 10 seconds

$$v(t) = f'(t) = 2t + \frac{2}{t^2} - \frac{6}{t^3} \Rightarrow f'(10) = 20 + \frac{2}{100} - \frac{6}{1000} = 20.0196$$

b) Find the acceleration after 10 seconds

$$a(t) = v'(t) = 2 - \frac{4}{t^3} + \frac{18}{t^4}$$

$$a(10) = 2 - \frac{4}{1000} + \frac{18}{10000}$$

c) When is the particle at rest (other than for $t = 0$)

when $v(t) = 0$, $v(t) = 2t + \frac{2}{t^2} - \frac{6}{t^3} = 0 \Rightarrow 0 \quad | \cdot t^3$

$$= 2t^4 + 2t - 6 = 0 \Rightarrow \textcircled{?}$$

Maple says: $t = -1.452$

d) When is the particle moving forward and when backward

forward if $v(t) > 0$ and backward if $v(t) < 0$
 too complicated to figure out by hand!

14. Find the following limits at infinity:

$$\lim_{x \rightarrow \infty} \frac{2x + 3x^4}{4x^3 - 2x^2 + x - 1} = \infty$$

$$\lim_{x \rightarrow \infty} \frac{x - x^5}{x^3 - x^2 + x - 1} = -\infty$$

$$\lim_{x \rightarrow \infty} \frac{4x^3 - 2x^2 + x - 1}{2x - 3x^4} = 0$$

$$\lim_{x \rightarrow \infty} \frac{x^3 - x^2 + x - 1}{x - 3x^3} = -\frac{1}{3}$$

$$\lim_{x \rightarrow \infty} \frac{(3x+4)(x-1)}{(2x+7)(4x+2)} = \frac{3}{8}$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 1}}{x} = \lim_{x \rightarrow \infty} \frac{x \sqrt{1 - \frac{1}{x^2}}}{x} = 1$$