Panel 1

Last Time.

$f(x) = e^x$ $\qquad f'(x) = e^x$

$f(x) = a^x$ $\qquad f'(x) = \ln(a)a^x$

$f(x) = \ln(x)$ $\qquad f'(x) = \frac{1}{x}$ $\qquad e^x = y \iff x = \ln(y)$

$f(x) = \log_5(x)$ $\qquad f'(x) = \frac{1}{\ln(5)x}$ $\qquad 5^x = y \iff x = \log_5(y)$

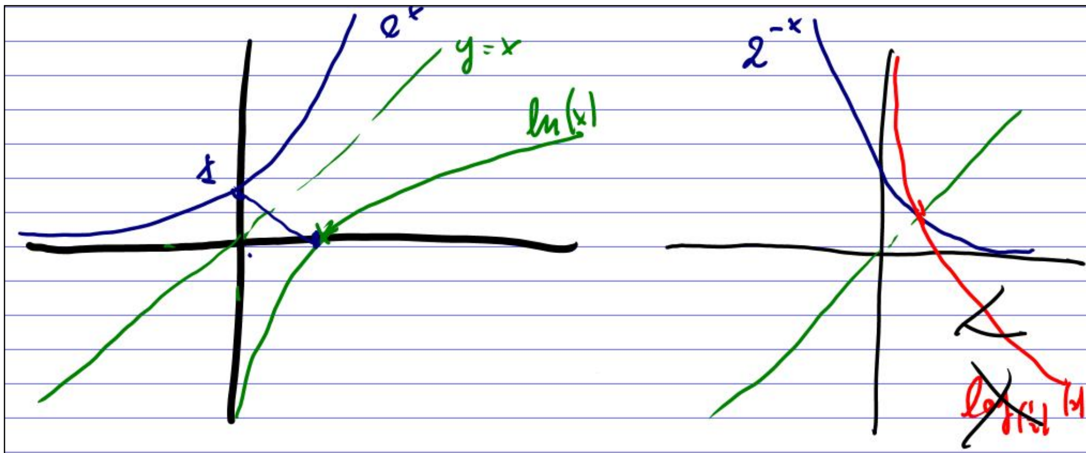
$y = \frac{x^5\sqrt{x-1}}{(x^2-2)^2}$ $\qquad \ln(y) = \ln(x^5\sqrt{x-1}) - 2\ln(x^2-2) =$

$= 5\ln(x) + \frac{1}{2}\ln(x-1) - 2\ln(x^2-2)$

Exp. growth / ~~decay~~:

$\frac{1}{y}\cdot y' = \frac{5}{x} + \frac{1}{2}\frac{1}{x-1} - 2\frac{1}{x^2-2}\cdot 2x$

Panel 2



Panel 3

EXAMPLE 2 The half-life of radium-226 ($^{226}_{88}Ra$) is 1590 years.
(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of $^{226}_{88}Ra$ that remains after t years.
(b) Find the mass after 1000 years correct to the nearest milligram.
(c) When will the mass be reduced to 30 mg?

a) $A(t) = 100e^{-0.000435t}$

b) $A(1000) = 100e^{-0.000435 \cdot 1000}$
$= 64$

Model: $A(t) = A_0 e^{kt}$

c) $A(t) = 30 = 100e^{-0.000435 \cdot t}$

A_0 = initial amount

k = growth/decay rate

$A_0 = 100$

$\Rightarrow A(t) = 100e^{kt}$ Know: $A(1590) = 50$
$= 100e^{k1590}$

$50 = 100e^{k \cdot 1590} \Rightarrow \frac{1}{2} = e^{k \cdot 1590}$ ln( )

$\ln(1) - \ln(2) = \ln(\frac{1}{2}) = \ln(e^{k \cdot 1590}) = k \cdot 1590 \Rightarrow k = \frac{-\ln(2)}{1590}$

Panel 4

$y = \sin(x) \cdot \ln(x)$.

$y' = \cos(x) \cdot \ln(x) + \frac{\sin(x)}{x}$

Panel 5

Inverse Trig Functions

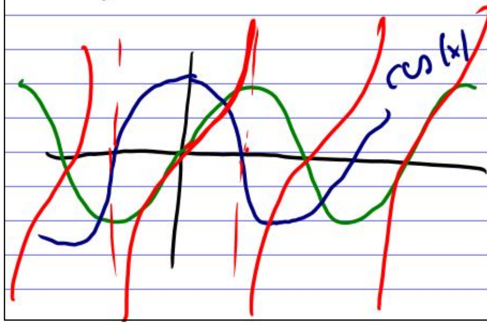
$\log_2(32) = y = 5 \mid 2^x$

$2^{\log_2(32)} = 32 = 2^y = 2^5$

Our study of inverse functions led us to the discovery of a new function: $y = e^x \to$

$y = \ln(x)$ new function!!!

Try again: Inverse of sin, cos, tan

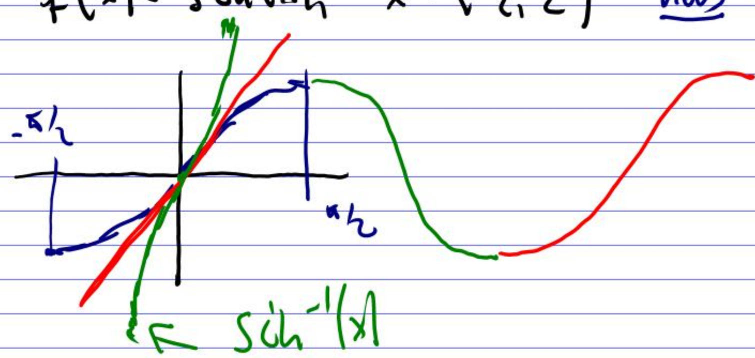


NONE have inverses, because none pass horizontal line test.

Panel 6

Inverse sin Function

Def: $f(x) = \sin(x)$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has inverse



Define $f(x) = \sin^{-1}(x) = \arcsin(x)$ as inverse of $\sin(x)$: $\sin^{-1}(x) = y \Leftrightarrow x = \sin(y)$

Panel 7

Ex $\sin^{-1}(0) = y$ / $\sin(\ )$ $\Rightarrow \sin^{-1}(0) = 0$

$\cancel{\sin}(\cancel{\sin^{-1}}(0)) = \sin(y)$

$0 = \sin(y)$ Guess: $y = 0$

$\sin^{-1}(\frac{1}{2}) = y = \frac{\pi}{6}$ / $\cdot\sin(\ )$

$\cancel{\sin}(\cancel{\sin^{-1}}(\frac{1}{2})) = \sin(y)$

$\frac{1}{2} = \sin(y) \to$

$\sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$, $\cos^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$

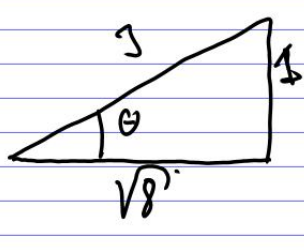
1	$\cos(0)$	$\sin(0)$	0
$\frac{\sqrt{3}}{2}$	$\cos(\frac{\pi}{6})$	$\sin(\frac{\pi}{6})$	$\frac{\sqrt{1}}{2}$
$\frac{\sqrt{2}}{2}$	$\cos(\frac{\pi}{4})$	$\sin(\frac{\pi}{4})$	$\frac{\sqrt{2}}{2}$
$\frac{1}{2}$	$\cos(\frac{\pi}{3})$	$\sin(\frac{\pi}{3})$	$\frac{\sqrt{3}}{2}$
0	$\cos(\frac{\pi}{2})$	$\sin(\frac{\pi}{2})$	$\frac{\sqrt{4}}{2} = 1$

Panel 8

We can do more sophisticated calculations:

Ex: $\tan(\sin^{-1}(1/3))$ Easy: $\cancel{\sin}(\cancel{\sin^{-1}}(\frac{1}{3})) = \frac{1}{3}$

$\sin^{-1}(\frac{1}{3}) = \theta$



Know: $\sin(\theta) = \frac{1}{3} = \frac{opp}{hyp}$

$\tan(\theta) = \frac{1}{\sqrt{8}}$

$\tan(\sin^{-1}(\frac{1}{3})) = \frac{1}{\sqrt{8}}$

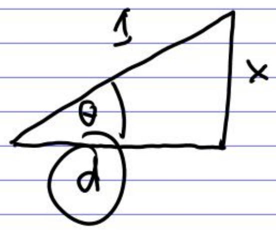
Panel 9

Derivative of $\sin^{-1}$

$y = \sin^{-1}(x) \iff \sin(y) = x \quad \Big| \frac{d}{dx}$

$\cos(y) \cdot y' = 1$

$\Rightarrow y' = \frac{1}{\cos(y)} = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}}$



$\sin^{-1}(x) = \theta \quad \Rightarrow \cos(\theta) = \frac{\sqrt{1-x^2}}{1}$

$\frac{x}{1} = \sin(\theta)$

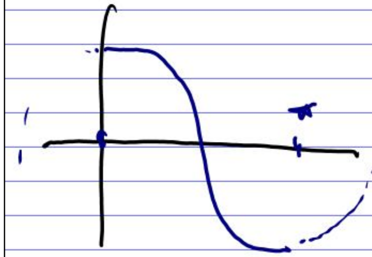
$d^2 + x^2 = 1^2 = 1 \Rightarrow d^2 = 1 - x^2 \Rightarrow d = \sqrt{1-x^2}$

Panel 10

Deriv. of $\cos^{-1}(x)$

Def: $f(x) = \cos(x)$, $x \in [0, \pi]$ has inverse:

$f^{-1}(x) = \cos^{-1}(x)$



Ex: $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$\frac{d}{dx} \cos^{-1}(x) = \frac{1}{-\sin(\cos^{-1}(x))}$

$f^{-1}(x) = \cos^{-1}(x)$

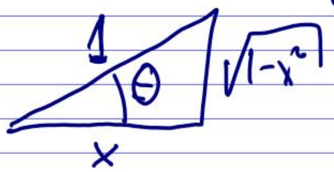
$\Rightarrow f(x) = \cos(x)$

$f'(x) = -\sin(x)$

Recall: $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

Panel 11

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sin(\cos^{-1}(x))} = -\frac{1}{\sqrt{1-x^2}}$$



$\cos^{-1}(x) = \theta$ $\quad |\cos(\ )$

$$\frac{x}{1} = \cos(\theta) = \frac{adj}{hyp}$$

$$\sin(\theta) = \frac{\sqrt{1-x^2}}{1}$$

$$\Rightarrow \sin(\cos^{-1}(x)) = \sqrt{1-x^2}$$

Panel 12

$f(x) = \cancel{\sin}^{-1}(x^2-1)$. Find f'

$$f'(x) = \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot 2x$$

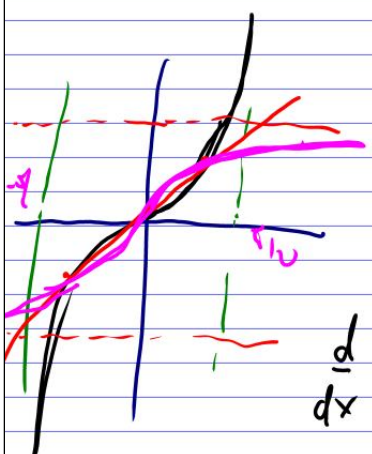
$g(x) = x\,\cancel{\cos}^{-1}(4x)$. Find g'

$$g'(x) = 1 \cdot \cos^{-1}(4x) + x \cdot \left(-\frac{1}{\sqrt{1-(4x)^2}}\right) \cdot 4$$

Panel 13

Inverse Tangent

$f(x) = \tan(x), \quad x \in (-\pi/2, \pi/2).$



$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$

$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$

$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{1+x^2}$

$\sec(\tan^{-1}(x)) = \sqrt{1+x^2}$ HW

Panel 14

Draw graph of $\tan(x)$ and $\tan^{-1}(x)$

See previous

If $f(x) = x \cdot \tan^{-1}(\sqrt{x})$, find $f'(x)$

$f'(x) = \tan^{-1}(\sqrt{x}) + x \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} =$

$\tan^{-1}(\sqrt{x}) + \frac{x}{2(1+x)\sqrt{x}}$

Panel 15

Summary

$\sin(x)$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$\cos(x)$, $x \in [0, \pi]$

$$\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$\tan(x)$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$$

Panel 16

l'Hospital's Rule

Nifty trick to compute complicated limits !!

Thm: Suppose f, g are diff'ble at $x=c$ and $\lim_{x\to c} f(x) = 0$, $\lim_{x\to c} g(x) = 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

applies if $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Also true if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$

Ex: $\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{\cos(0)}{1} = \frac{1}{1} = 1$

Panel 17

Ex: l'Hospital's Rule is perfect for tricky limits

$$\lim_{x\to 0} \frac{x^2-9}{x-3} = \frac{-9}{-3} = 3$$

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \frac{2\cdot(3)}{1} = 6 \overset{\text{new}}{} \quad \overset{\text{old}\to}{=} \lim_{x\to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = 6$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{\cos(0)}{1} = 1$$

$$\lim_{x\to 0} \frac{e^{x^2}-1}{x^3} = \lim_{x\to 0} \frac{2\cancel{x}e^{x^2}}{3x^{\cancel{2}}} = \lim_{x\to 0} \frac{2}{3}\frac{e^{x^2}}{x} = \frac{1}{0} = \text{undef}$$

Panel 18

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \frac{0}{0} = \lim_{x\to 1} \frac{1/x}{1} = 1$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{e^x}{2x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{3x^{2/3}}{x} = \lim_{x\to\infty} \frac{3}{x^{1/3}} = 0$$

$$\lim_{x\to 0^+} x\ln(x) = ?$$

Quiz: inverse trigs, l'Hospital's Rule

Panel 19

Graph $f(x) = xe^{-x^2}$

(1) Domain
(2) Asymptotes
(3) Critical pts:
(4) Infl points.
(5) Sign table
(6) Values
(7) Graph