Panel 1

Last time

Inverse functions: $\cancel{f}(\cancel{f^{-1}}(x)) = x$, $\cancel{f^{-1}}(\cancel{f}(x)) = x$

- if f is one-to-one
- if graph of f passes horizontal line test

$\sqrt{x^2} = |x|$

- $\frac{d}{dx} f^{-1}(x) = 1/f'(f^{-1}(x))$

(Natural) exp. function:

$f(x) = e^x$, $e = 2.71\ldots = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$\frac{d}{dx} e^x = e^x$, "deriv. of exp is itself"

Panel 2

Find inverse of $y = x^3+1$, and it's derivative

$y = x^3+1 = f(x)$.

$\frac{d}{dx} f^{-1} = 1/f'(f^{-1}(x))$

$y-1 = x^3$

$\sqrt[3]{y-1} = x$

$f^{-1}(x) = \sqrt[3]{x-1} = (x-1)^{1/3} \Rightarrow (f^{-1})'(x) = \frac{1}{3}(x-1)^{-2/3}$

$(f^{-1})'(x) = \frac{1}{3(\sqrt[3]{x-1})^2} = \frac{1}{3}(x-1)^{-2/3}$

$f'(x) = 3x^2$

$\frac{d}{dx} \cos(e^x)(e^{3x}) =$

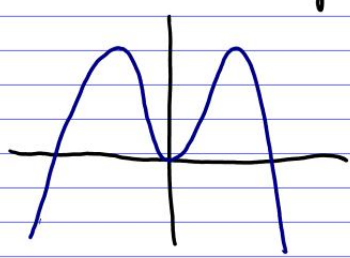
$-\sin(e^x)\, e^x \cdot e^{3x} + \cos(e^x)\cdot e^{3x}\cdot 3$

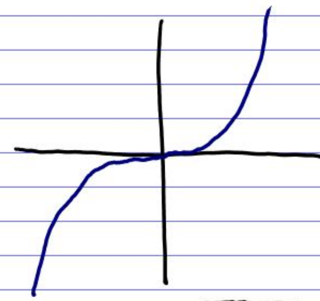
Panel 3

Name: ______

Quiz #8

① Which of the following function has an inverse?





② Find the inverse function for $f(x)=\sqrt{2x-1}$

Panel 4

③ Find the given derivatives:

a) $f(x)=5e^{3x^2}$; $f'(x)=5e^{3x^2}(6x)$

b) $f(x)=\frac{(e^x-e^{-x})}{x}$; $f'(x)=\frac{(e^x+e^{-x})x-(e^x-e^{-x})\cdot 1}{x^2}$

c) $f(x)=e^{2x}$; $f'''(x)=$

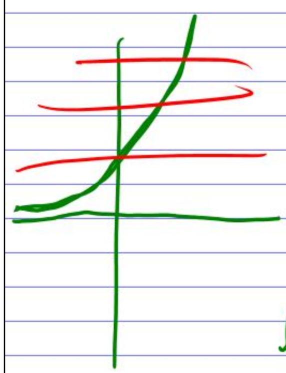
$f'(x)=2e^{2x}$, $f''(x)=2\cdot 2e^{2x}$, $f'''(x)=2^3e^{2x}$

$f^{(10)}(x)=2^{10}e^{2x}$

Panel 5

<u>Thm.</u> Since $f(x)=e^x$ passes horizontal line test, it has an inverse, which is continuous and has as derivative

$$f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{x}$$



$e^x = f(x)$, $e^{f^{-1}(x)} = f(f^{-1}(x)) = x$

<u>Def</u> $\ln(x)$ ~~b~~ is the inverse function of e^x. Know: $\frac{d}{dx}\ln(x) = \frac{1}{x}$

$\ln(e^2) = 2$ "Deriv. of ln is 1 over"

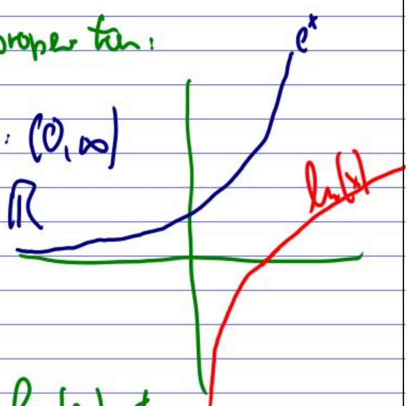
Panel 6

<u>Natural logarithm</u> $\ln(x)$ has properties:

domain of $\ln(x)$ (= range of e^x): $(0,\infty)$

range of $\ln(x)$ (= domain of e^x): $\mathbb{R}$

$\frac{d}{dx}\ln(x) = \frac{1}{x}$



(0) $e^{\ln(x)} = x$ $\ln(e^x) = x$, $\underline{\ln(e)=1}$

(1) $\ln(x^p) = p\ln(x)$

(2) $\ln(x\cdot y) = \ln(x) + \ln(y)$

(3) $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

Panel 7

Ex: Find $\frac{d}{dx} 5^x = \frac{d}{dx} e^{\ln(5^x)} = \frac{d}{dx} e^{x \cdot \ln(5)} = e^{x \ln(5)} \cdot \ln(5)$

$= 5^x \cdot \ln(5)$

Find $\frac{d}{dx} a^x = a^x \ln(a)$ $\left(\frac{d}{dx} e^x = e^x \overbrace{\ln(e)}^{1} = e^x \right)$

Def: $\log_5(x)$ is inverse of 5^x. It is diff'ble and $\frac{d}{dx} \log_5(x) = \frac{1}{5^{\log_5(x)} \ln(5)} = \frac{1}{\ln(5) \cdot x}$

$= 1 / f'(f^{-1}(x))$ $\left(\frac{d}{dx} \ln(x) = \frac{d}{dx} \log_e(x) = \frac{1}{\ln(e)\, x} = \frac{1}{x} \right)$

Panel 8

New Derivatives:

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} a^x = \ln(a) a^x$

$\frac{d}{dx} \ln(x) = \frac{1}{x}$

Memorize ✓ Me

$\frac{d}{dx} \log_5(x) = \frac{1}{\ln(5)\, x}$

Panel 9

Ex: Find the following derivatives

a) $f(x) = x^2 \ln(x-1)$ $\Rightarrow$

$f'(x) = 2x \cdot \ln(|x-1|) + x^2 \cdot \frac{1}{x-1} \cdot 1$

$= -\tan(x)$

b) $g(x) = \ln(\cos(x))$ $\quad g'(x) = \frac{1}{\cos(x)}(-\sin(x)) = -\frac{\sin(x)}{\cos(x)} =$

c) $h(x) = e^{x^2} \ln\left(\sqrt{1-x^2}\right)$, $h' = e^{x^2} 2x \ln\left(\sqrt{1-x^2}\right) +$

$(1-x^2)^{\frac{1}{2}}$

$= e^{x^2} \ln\left[(1-x^2)^{1/2}\right] =$ $\quad + e^{x^2} \frac{1}{\sqrt{1-x^2}} \frac{1}{2} (1-x^2)^{-1/2} \cdot (-2x) =$

$= e^{x^2} \cdot \frac{1}{2} \ln(1-x^2)$ $\quad + e^{x^2} \frac{(-x)}{1-x^2}$

Panel 10

Differentiate $y = \frac{x^{3/4} \cdot \sqrt{x^2+1}}{(3x+2)^5}$ Crazy!!!!

Trick: $\ln(y) = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right) =$ $\quad \Big| \frac{d}{dx}$

logarithmic diff

$= \ln\left(x^{3/4}\sqrt{x^2+1}\right) - \ln\left((3x+2)^5\right) =$

$= \ln\left(x^{3/4}\right) + \ln\left(\sqrt{x^2+1}\right) - 5\ln(3x+2) =$

$= \frac{3}{4}\ln|x| + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$

$\frac{d}{dx}\ln(y) = \frac{1}{y} \cdot y' = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} - 5\frac{3}{3x+2}$ $\quad y$

$y' = \left(\frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} - \frac{15}{3x+2}\right) \cdot \left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)$

Panel 11

$$y = \frac{x\sqrt{x-1}}{(3x+2)^2}$$ Find $y' = ?$ for und

$$\ln|y| = \ln\left(\frac{x\sqrt{x-1}}{(3x+2)^2}\right) = \ln\left(x\sqrt{x-1}\right) - \ln\left((3x+2)^2\right)$$

$$= \ln(x) + \ln\left(\sqrt{x-1}\right) - 2\cdot\ln(3x+2) =$$

$\frac{d}{dx}$ $\quad d/dx$

$$= \ln|x| + \frac{1}{2}\ln|x-1| - 2\ln(\underbrace{3x+2})$$

$$\Rightarrow \frac{y'}{y} = \frac{1}{x} + \frac{1}{2}\frac{1}{x-1} - \frac{2}{3x+2}\cdot 3 \Rightarrow$$

$$y' = \left(\frac{1}{x} + \frac{1}{2(x-1)} - \frac{6}{3x+2}\right)\overbrace{y} = \left(\frac{1}{x} + \frac{1}{2(x-1)} - \frac{6}{3x+2}\right)\boxed{\frac{x\sqrt{x-1}}{(3x+2)^2}}$$

Panel 12

Exp. Growth + Decay

Law of natural growth or decay:

" rate of change ~~depends on~~ is proportional to how much is there." $\Rightarrow$

Ex: Population at t is $P(t)$.

$P'(t) = k\,P(t)$

if $P(t)$ is small $\Rightarrow$ small rate of change

$\boxed{\text{Solution} \quad P(t) = P_0 e^{kt}}$

if $P(t)$ is large $\Rightarrow$ large rate of change

Panel 13

EXAMPLE 1 Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.)

What is the population in 2020?

$P(t) = P_0 e^{kt}$, t = time since 1950, e.g. $t=10$ in 1960

$t=70$ in 2020

$P(0) = 2560 = P_0$ $\quad$ P_0 is initial pop., i.e. $P(0) = 2560$

k = growth rate

$P(10) = 3040 = P_0 e^{k \cdot 10}$

$\Rightarrow 3040 = 2560 e^{k10} \Rightarrow \frac{3040}{2560} = e^{k \cdot 10}$ $\quad$ $\ln()$

$\Rightarrow \ln\left(\frac{3040}{2560}\right) = \cancel{\ln}\left(\cancel{e}^{k \cdot 10}\right) = k \cdot 10$

$\Rightarrow k = \frac{1}{10} \ln\left(\frac{3040}{2560}\right) = \underline{0.01719}$

Panel 14

$P(t) = 2560 e^{0.01719t}$

$P(70) = 2560 e^{0.01719 \cdot 70} = \underline{8524}$ Million people in