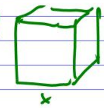
Panel 1

Last time

Related Rates — surface area of cube. Length of side changes at rate of 2 cm/sec. How about surface area when S = 100?



$S = 6x^2 \quad \left| \frac{d}{dt} \right.$

Linearization

$S' = 12x \cdot x' = 12 \cdot 4.08 \cdot 2$

Want: S' Know: x'

Differentials

$100 = 6x^2 \quad x = \sqrt{\frac{100}{6}} = 4.08$

Error Estimation + relative Error

Panel 2

Linearization $f(x) \approx f'(c)(x-c) + f(c)$

Compute f', find $f'(c)$, find $f(c)$

$\tan(x)$ near zero. $f'(x) = \sec^2(x)$

$c = 0$: $f'(0) = \sec^2(0) = 1$

$f(0) = 0$

$\tan(x) \approx 1 \cdot (x - 0) + 0 = x$

$\Rightarrow \tan(x) \approx x$

Thus: $\tan(0.1) \approx 0.1$

Panel 3

Error Estimate: measure length of side of a cube as 5 cm ± 0.2 cm.

Error in computing surface area?

$S = 6x^2$

$\frac{dS}{dx} = 12x \Rightarrow dS = 12x \cdot dx$

$S = 625$ [sic: $S = 150$]

$dx = \pm 0.2, \ x = 5$

$\Rightarrow dS = 12 \cdot 5 \cdot 0.2 = 60 \cdot 0.2 = \underline{12}$ 0.8%

Relative Errors: $\frac{dx}{x} = \frac{0.2}{5} = 0.04$ or 4% $\quad \frac{dS}{S} = \frac{12}{150} = 0.008$ or

Panel 4

Volume of cube is 6 cm ± 0.1. What is

a) error in volume? 10.8

b) relative error in volume? $\frac{10.8}{216} = 0.05$ or 5%

$V = x^3 \Rightarrow dV = 3x^2\, dx$

$= 3 \cdot 6^2 \cdot 0.1 = \underline{10.8}$

$V = 6^3 = \underline{216}$

rel error of x is $\frac{0.1}{6} = 0.018$ or 1.6%

Panel 5

Quiz #2 Name: ______

① Find the linearization of $f(x) = \sqrt{2+x}$ near $c = 2$ and use it to estimate $\sqrt{4.1}$

② The radius of a disk is measured to be 24 cm
Estimate the error in computing area of the di

Panel 6

③ Two cars start from the same spot. One travels
other goes north at 25 mph. At what rate is the
the two cars increasing two hours later ?

Panel 7

Inverse Functions

Certain functions cancel each other out:

$\sqrt{x^2} = x \Rightarrow (\sqrt{x})^2 = x \qquad f(x) = x^2, \; f^{-1}(x) = \sqrt{x}$

$\frac{1}{3} 3x = x$

Such pairs of functions are called inverse of each other:

Def: f and g are inverse of each other if

$f(g(x)) = x \qquad g(f(x)) = x$

Note: One of these functions is usually written

as: $f^{-1}(x) \qquad f(f^{-1}(x)) = x, \; f^{-1}(f(x)) = x$

Panel 8

Questions:

Which functions have an inverse function?

How do you find the inverse function for $f(x)$?

If f has property X, does f^{-1} have it, too?

continuity

derivative

Panel 9

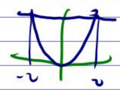
Thm: A function f has an inverse f^{-1} if:

a) f has inverse if f is one-to-one,

if $f(x_1)=f(x_2) \Rightarrow x_1=x_2$ Ex: $f(x)=x^2$

one-to-one No!

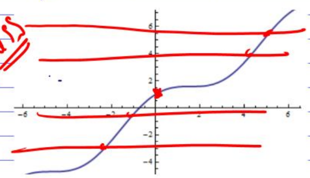
b) The graph of f passes horizontal line test



Ex: Which function has an inverse? Find $f^{-1}(1)=0$

$f(x)=x+\cos(x)$ $f^{-1}(1)=b$ if $g(x)=\sin(x)$

pass

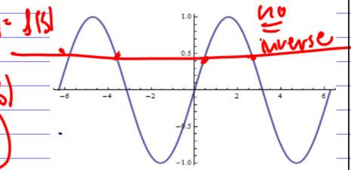


$f(f^{-1}(1))=f(b)$

$1=f(b)$

$b=0$

no inverse



Panel 10

Ex: If $f(x)=\frac{x+2}{x-3}$, $x\neq 3$, find $f^{-1}(x)$.

Graph f: domain, asympt., critical, incr/decr, concavity, ...

$y=\frac{x+2}{x-3}$ Set $f(x)=y$

$(x-3)y=x+2$ Solve for x

$xy-3y=x+2$ Switch x and y.

$xy-x=2+3y$

$x(y-1)=2+3y$ $f^{-1}(x)=\frac{3x+2}{x-1}$

$x=\frac{3y+2}{y-1} \Leftrightarrow y=\frac{3x+2}{x-1}$ $f(x)=\frac{x+2}{x-3}$

Panel 11

Thm: If f is continuous, then

Thm If f is differentiable, then

Ex: If $f(x)=x^2$, find $\frac{d}{dx} f^{-1}(x)$

Panel 12

Is $f^{-1}(x)=\frac{3x+2}{x-1}$ inverse of $f(x)=\frac{x+2}{x-3}$?

$$f(f^{-1}(x))=\frac{\frac{3x+2}{x-1}+2}{\frac{3x+2}{x-1}-3}\cdot\frac{(x-1)}{(x-1)}=\frac{3x+2+2(x-1)}{3x+2-3(x-1)}=\frac{5x}{5}=x$$

Yes, they are ✓

Panel 13

Want to apply f^{-1}- strategy to new functions.

Exponential Function: $f(x) =$

Panel 14

Thm: If f is continuous, then f^{-1} is continuous

Thm If f is differentiable, then f^{-1} is also.

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Ex: If $f(x) = x^2$, find $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

$f^{-1}(x) = \sqrt{x}$

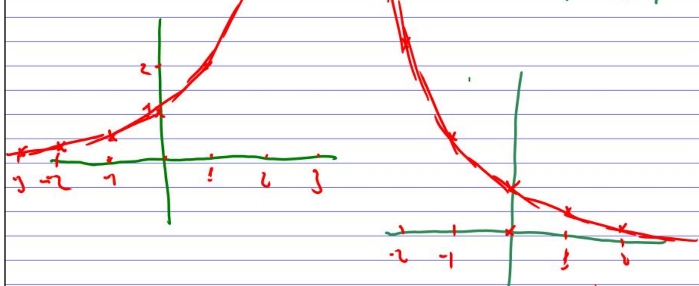
$$= \frac{1}{2\cdot(\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

Panel 15

Want to apply f^{-1}- strategy to new functions.

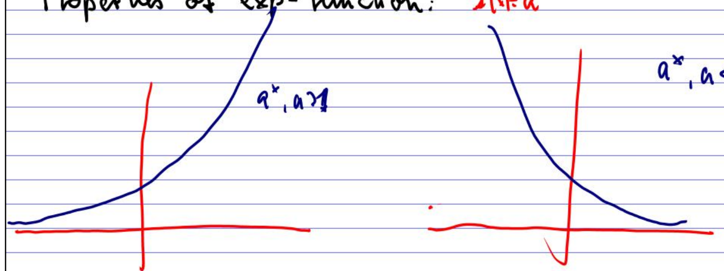
Exponential Function: $f(x) = a^x$, $a > 0$

$f(x) = 2^x$, $g(x) = \left(\frac{1}{2}\right)^x$



Panel 16

Properties of exp- function: $f(x) = a^x$



$\lim_{x\to\infty} a^x = \infty$ $\qquad$ $\lim_{x\to\infty} a^x = 0$

$\lim_{x\to-\infty} a^x = 0$ $\qquad$ $\lim_{x\to-\infty} a^x = \infty$

Panel 17

The natural exp. function: $f(x) = e^x$, why e???

$e = 2.7129... = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ $g(x) = 2^x$ $\boxed{\text{easiest}}$

Euler's #

$h(x) = \left(\frac{1}{3}\right)^x$

$f(2) = e^2 \approx 7.3$ What is $\frac{d}{dx} e^x = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$g(3) = 8$ $= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} =$

$h(2) = \frac{1}{9}$

$= \lim \frac{e^x e^h - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}$

Panel 18

$e = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{h\to 0} (1+h)^{1/h}$

$h = \frac{1}{x}$

$e \approx (1+h)^{1/h}$ $(\ )^h$ $\Rightarrow e^h \approx 1+h$

$\Rightarrow \frac{e^h - 1}{h} = \frac{1+h-1}{h} = \frac{h}{h} = 1$

$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

$\frac{d}{dx} e^x = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1$

Theorem: $\frac{d}{dx} e^x = e^x$ That's why it is natural exp. function

Panel 19

Ex: Find the derivatives of

a) $f(x) = 5e^{3x^2}$ $f'(x) = 5e^{3x^2} \cdot 6x$

$\frac{d}{dx} \sin(3x^2) = \cos(3x^2) \cdot 6x$

b) $g(x) = \frac{e^{2x}}{x^2}$ $g'(x) = \frac{e^{2x} \cdot 2 \cdot x^2 - e^{2x} \cdot 2x}{(x^2)^2}$

c) $h(x) = \sin(x) \cdot e^{3x}$ $h'(x) = \cos(x) \cdot e^{3x} + \sin(x) \cdot e^{3x} \cdot 3$

d) $k(x) = e^{\cos(x^2)}$ $\Rightarrow k'(x) = e^{\cos(x^2)}(-\sin(x^2)) \cdot 2x$

Panel 20

Derivative of exp. is itself