Panel 1

Last Time: consider $x^2 + xy = 4 \Rightarrow xy = 4 - x^2$

$y = \frac{4-x^2}{x}$

a) $y = y(x)$, $y' = ?$

$\frac{d}{dx}(x^2 + xy) = \frac{d}{dx}(4)$

$2x + y + xy' = 0 \Rightarrow xy' = -2x - y$

$y' = \frac{-2x-y}{x}$

b) $x = x(y)$

x'

$\frac{d}{dy}(x^2 + xy) = \frac{d}{dy}(4)$

$2x \cdot x' + x' y + x = 0$

c) $x = x(t)$, $y = y(t)$

Find x' and y'

$\frac{d}{dt}(x^2 + xy) = \frac{d}{dt}(4) = 0$

$\boxed{2xx' + x'y + x \cdot y' = 0}$

Panel 2

Appl. of Implicit Differentiation

Ex: Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm³/sec. How fast is the radius increasing if the diameter is 50 cm.

$r = 25$

$V = \frac{4}{3}\pi r^3$

$V = V(t)$, $r = r(t)$

Want: $\frac{dr}{dt} = r'(t)$.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$

want

Know $V' = 100$

$V' = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \cdot r'$

???

$100 = 4\pi (25)^2 \cdot r'$

$\Rightarrow r' = \frac{100}{4 \cdot \pi \cdot 25^2} = \frac{25}{\pi \cdot 25^2} = \frac{1}{\pi \cdot 25} \approx 0.0127$

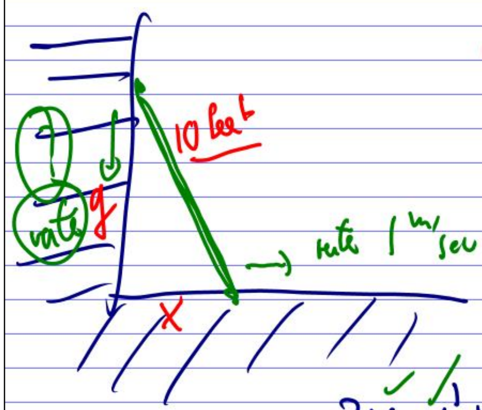
Panel 3

Ex: A 10 foot ladder rests against a wall. The base of the ladder slides away from the wall at a rate of 1 ft/sec. How fast is the top sliding down when the bottom is 6 feet away from the wall.

$x = 6: 36 + y^2 = 100$

$y^2 = 64$

$y = 8$



Want: $\frac{dy}{dt} = y'$

Know: $\frac{dx}{dt} = x' = 1$

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(100)$$

$$\boxed{y' = -\frac{12}{16}}$$

$$2x \cdot x' + 2y \cdot y' = 0 \Rightarrow 2 \cdot 6 \cdot 1 + 2 \cdot 8 \cdot y' = 0$$

$$12 + 16y' = 0$$

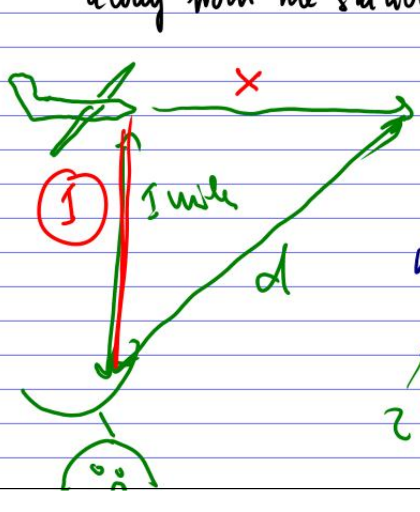
Panel 4

Ex: A plane flying horizontally at an altitude of 1 mile and a speed of 500 mph directly over a radar station. Find the rate at which the distance from plane to radar station is increasing when it is 2 miles away from the station.

$d^2 = x^2 + 1$

$2^2 = x^2 + 1 \Rightarrow 3 = x^2$

Speed 500



Want: $d' = ?$

Know: $d^2 = x^2 + 1$ $\Big| \frac{d}{dt}$

Want

$$2d \cdot d' = 2x x'$$

$d = 2$, $x = \sqrt{3}$, $x' = 500$

$$d' = \frac{\sqrt{3} \cdot 500}{2}$$

Panel 5

Linearization

Recall: $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$

$\Rightarrow f(x) - f(c) \approx f'(c)(x-c)$

$\Rightarrow f(x) \approx f'(c)(x-c) + f(c)$ is called

linearization of $f(x)$

Ex: Find linearization of $f(x) = \sqrt{x+3}$ at $x=1$

$\Rightarrow \sqrt{x+3} \approx f'(c)(x-c) + f(c)$ $f' = \frac{1}{2}(x+3)^{-1/2} \cdot 1$

$= \frac{1}{2}(1+3)^{-1/2}(x-1) + f(1) = \frac{1}{4}(x-1) + \sqrt{1+3}.$

Panel 6

$\sqrt{x+3} \approx \frac{1}{4}(x-1)+2$

Use linearization of $\sqrt{x+3}$ to compute $\sqrt{3.98}$

Know $\sqrt{3.99} = \sqrt{0.99+3} \approx \frac{1}{4}(0.99-1)+2$

$= \frac{-0.01}{4} + 2 = 0.005 + 2 = 1.995$

Linearization is useful to estimate hard-to-compute values of some functions

Panel 7

Ex. Find $\sin(0.05)$ (=?)

Try to use linearization:

$$\sin(x) - \sin(c) \approx f'(c)(x-c)$$

$$\sin(x) \approx f'(c)(x-c) + \sin(c) \qquad \text{Pick } c = 0$$

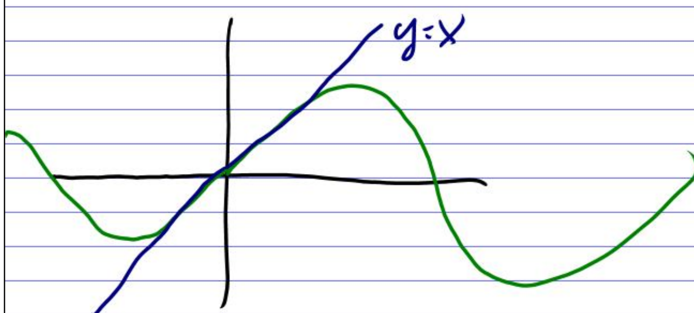
$$\cancel{\sin(x) \approx \cos(0)}(x-0) + \sin(0)$$

$$\sin(x) \approx x \qquad \boxed{\text{if } x \text{ is close to } 0}$$

$\Rightarrow \sin(0.05) \approx 0.05$ ($\sin(0.05) = 0.04997..$)

$\sin(-0.02) \approx -0.02$ ✓ $\qquad \sin(2) \approx 2$

Panel 8



Linearisation allows you to replace difficult functions by simple ones (linear ones)

Panel 9

Differentials If $y = f(x)$

The differential dy is defined as

$$\frac{dy}{dx} = f'(x) \qquad \boxed{dy = f'(x)\,dx}$$

Ex: $f(x) = \sqrt{x+3}$. Find dy if $x=1$ and $dx = 0.05$

$dy = f'(1)\,dx = f'(1) \cdot 0.05$

$\frac{1}{2}(x+3)^{-1/2}$ at $x=1$: $\frac{1}{4}$

$\Rightarrow \boxed{dy = \frac{1}{4}\,dx = \frac{1}{4} \cdot 0.05}$

Panel 10

Error Estimation: The radius of a sphere was measured as 2 cm with an error of 0.05 cm. What is the impact of this error if radius is used to compute volume of the sphere.

perfect volume $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\, 2^3$

$dV = V'(r) \cdot dr$

$\Rightarrow dV = \frac{4}{3}\pi\, 3r^2\, dr = 4\pi r^2\, dr = 4\pi r^2 \cdot 0.05 =$

$= 4 \cdot \pi\, (2)^2\, 0.05 = \boxed{2.51}$

Panel 11

Summary of Applications of Derivatives

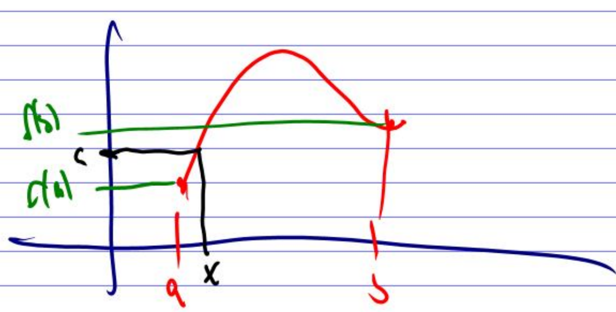
1. Curve sketching
2. Optimization (max/min)
3. Implicit Diff
4. Related Rates
5. Linearization
6. Error estimates

Quiz on Wed.

Panel 12

Final Application: Mean Value Theorem

Recall IVT (Intermediate Value Theorem): if f is cont. on $[a,b]$, and c is a number between $f(a)$ and $f(b)$, then there is x s.t. $f(x)=c$



Panel 13

MVT: If f is differentiable on (a,b) and cont. on $[a,b]$, then

$$\frac{f(b)-f(a)}{b-a} = f'(c), \quad c \in (a,b)$$

slope